

THE EQUIVALENCE PROBLEM FOR 5-DIMENSIONAL LEVI DEGENERATE CR MANIFOLDS

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ABSTRACT. Let M be a CR manifold of hypersurface type, which is Levi degenerate but also satisfying a k -nondegeneracy condition at all points. This might be only if $\dim M \geq 5$ and if $\dim M = 5$, then $k = 2$ at all points. We prove that for any 5-dimensional, uniformly 2-nondegenerate CR manifold M there exists a canonical Cartan connection, modelled on a suitable projective completion of the tube over the future light cone $\{z \in \mathbb{C}^3 : (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, x^3 > 0\}$. This determines a complete solution to the equivalence problem for this class of CR manifolds.

1. INTRODUCTION

Let M be a 5-dimensional CR hypersurface, which is Levi degenerate at all points. Quite simple examples are provided by Cartesian products of the form $M = \overline{M} \times \mathbb{C}$ for some 3-dimensional CR manifold \overline{M} . A much less trivial case is represented by the so-called *tube over the future light cone*

$$\mathcal{T} = \{z \in \mathbb{C}^3 : (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, x^3 > 0\} \subset \mathbb{C}^3. \quad (1.1)$$

This hypersurface is in fact Levi degenerate at all points (it is homogeneous w.r.t. $\text{Aut}(\mathcal{T})$), it is foliated by complex leaves and yet it admits no local *CR straightening*, that is no local CR equivalence with a Cartesian product of the form $\overline{M} \times \mathbb{C}$.

Freeman ([Fr]) found necessary and sufficient conditions for a real analytic CR manifolds to admit local straightenings, together with obstructions to the existence of CR straightenings in the smooth category. Such obstructions are equivalent to the so-called *k -nondegeneracy conditions* at its points ([BER, KZ]). We recall that a CR hypersurface M satisfies the 1-nondegeneracy condition at all points if and only if it is Levi nondegenerate and that the other k -nondegeneracy conditions for $k \geq 2$ can be taken as progressively weaker nondegeneracy conditions.

The smallest possible dimension for a CR hypersurface M to be Levi degenerate and yet k -nondegenerate at all points is 5. In such a case k is necessarily equal to 2. For brevity, we call the 5-dimensional, 2-nondegenerate CR hypersurfaces of uniform type *girdled CR manifolds*.

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These CR manifolds have been recently considered in several studies, which for instance brought to Ebenfelt's normal forms for real analytic, Levi degenerate CR hypersurfaces in \mathbb{C}^3 and Kaup and Fels' classification of homogeneous Levi degenerate 5-dimensional CR manifolds ([Eb, FK1]).

In this paper, we give a general solution to the equivalence problem for girdled CR manifolds in the \mathcal{C}^∞ category, proving the existence of a canonical Cartan connection for any such manifold M .

We recall that (see e.g. [Ko, Sh]) a *Cartan connection on a manifold N , modelled on a homogeneous space G/H* , is a pair (Q, ϖ) , formed by a principal H -bundle $\pi : Q \rightarrow N$ and a \mathfrak{g} -valued 1-form $\varpi = TQ \rightarrow \mathfrak{g} = \text{Lie}(G)$ satisfying the following conditions:

- (a) $\varpi_y : T_y Q \rightarrow \mathfrak{g}$ is a linear isomorphism for any $y \in Q$ and

$$(\varpi_y)^{-1}|_{\mathfrak{h}} : \mathfrak{h} = \text{Lie}(H) \rightarrow T_y^{\text{Vert}} Q$$

is the standard isomorphism given by the action of H on Q ,

- (b) $R_h^* \varpi = \text{Ad}_{h^{-1}} \varpi$ for any $h \in H$.

If N is a manifold endowed with a fixed geometric structure, a Cartan connection (Q, ϖ) is said *canonical* if there exists a natural correspondence between the automorphisms of the geometric structure and the diffeomorphisms $\hat{f} : Q \rightarrow Q$ such that $\hat{f}^* \varpi = \varpi$. We point out that, if (Q, ϖ) is a canonical Cartan connection, any fixed basis (E_i^o) of \mathfrak{g} gives a canonical *absolute parallelism* on Q (also called *$\{e\}$ -structure*), namely the collection of vector fields

$$E_i|_y = \varpi_y^{-1}(E_i^o), \quad y \in Q.$$

Since the structure functions of a canonical absolute parallelism give a complete set of invariants for the geometric structure on N (see e.g. [St, AS]), for the geometric structures that admit canonical Cartan connections, the equivalence problems have exact and complete solutions.

Canonical Cartan connections give also valuable information on the automorphism groups of the considered geometric structures and allow constructions of useful special coordinates (see e.g. [Ko, Sh, SS1]).

Our main result on girdled CR manifolds is the following:

Theorem 1.1. *For any 5-dimensional girdled CR manifold (M, \mathcal{D}, J) , there exists a canonical Cartan connection (Q, ϖ) , modelled on the projective completion $M_o = \text{SO}_{3,2}^0/H \subset \mathbb{CP}^4$ of \mathcal{T} (see §3 for definition of M_o).*

The proof is constructive and provides an explicit description of the bundle $\pi : Q \rightarrow M$ and of the \mathfrak{g} -valued 1-form ϖ . Roughly speaking, it consists of a suitable modification of Tanaka's construction of Cartan connections for geometric structures modelled on semi-simple Lie groups ([Ta3, AS]). In particular, our result can be taken as the analogue of the Cartan connections of Levi nondegenerate hypersurfaces (Tanaka's and Chern-Moser's connections) and of the CR manifolds endowed with the so-called *parabolic geometries* ([Ta2, Ta3, CM, SS1, SS, SS1]).

Note that, since the isotropy of the model space $M_o = \mathrm{SO}_{3,2}^0/H$ is not a parabolic subgroup, the girdled CR structures cannot be classified as parabolic geometries. On the other hand, our modifications of Tanaka's approach can be very likely extended to other dimensions and other types of finitely nondegenerate CR structures and we expect the existence of an interesting new class of geometries, which includes both girdled CR structures and parabolic geometries as special cases.

We conclude mentioning that another absolute parallelism for girdled CR manifolds was previously determined by Ebenfelt in [Eb1]. However, such parallelism can be considered as canonical only if one consider CR automorphisms satisfying certain additional assumptions. It consequently provides only solutions to the corresponding restricted equivalence problem.

After finishing our paper, we realised that Isaev and Zaitsev recently constructed an absolute parallelism for girdled CR manifolds, in general not determined by a Cartan connection, which therefore gives an alternative solution to the equivalence problems for such manifolds ([IZ]).

The paper is organised as follows: in §2 and §3 we give the basic definitions and properties of girdled CR manifolds, of the tube \mathcal{T} over the future light cone and of its projective completion $M_o \subset \mathbb{CP}^4$; in §4, we introduce some definition and simple facts on vector spaces with filtrations, which will be used in later sections; in §5, §6 and §7, we construct the three steps of a tower, which is canonically associated with a girdled CR manifold and is the analogue of Tanaka's tower of Levi-nondegenerate CR manifolds; in §8, we prove the main theorem and determine the structure equations of a girdled manifold.

Notation. In the following, for any given (real or complex) subbundle $\mathcal{K} \subset T^{\mathbb{C}}M$ of the complexified tangent space $T^{\mathbb{C}}M$, we indicate by $\underline{\mathcal{K}}$ the class of all local smooth section of \mathcal{K} .

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2. PRELIMINARIES

2.1. Finitely non-degenerate CR manifolds of dimension 5.

Given a $2n + k$ -dimensional manifold M , a *CR structure on M of codimension k* is a pair (\mathcal{D}, J) , formed by a distribution $\mathcal{D} \subset TM$ of codimension k and a smooth family of complex structure $J_x : \mathcal{D}_x \longrightarrow \mathcal{D}_x$ satisfying the following integrability condition:

the bundle $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$, given by the $+i$ -eigenspaces $\mathcal{D}_x^{10} \subset \mathcal{D}_x^{\mathbb{C}}$ of the complex structure J_x , is involutive, i.e.,

$$[X^{10}, Y^{10}] \in \underline{\mathcal{D}}^{10} \quad \text{for any pair } X^{10}, Y^{10} \in \underline{\mathcal{D}}^{10}.$$

The complex vector bundles \mathcal{D}^{10} , $\mathcal{D}^{01} = \overline{\mathcal{D}^{10}}$ are called *holomorphic and anti-holomorphic bundles* of the CR structure (\mathcal{D}, J) . Two CR manifolds

(M, \mathcal{D}, J) and (M', \mathcal{D}', J') are called *(locally) CR equivalent* if there exists a (local) diffeomorphism $f : M \rightarrow M'$ such that

$$f_*(\mathcal{D}) = \mathcal{D}' , \quad f_*(J) = J' .$$

The *Freeman sequence* of a CR manifold (M, \mathcal{D}, J) ([Fr], Thm. 3.1) is the nested sequence of families of complex vector fields

$$\cdots \subset \underline{\mathcal{F}}_{j+1} \subset \underline{\mathcal{F}}_j \subset \cdots \subset \underline{\mathcal{F}}_1 \subset \underline{\mathcal{F}}_0 \subset \underline{\mathcal{F}}_{-1} = \underline{\mathcal{D}}^{\mathbb{C}}$$

iteratively defined by

$$\underline{\mathcal{F}}_{k+1} = \underline{\mathcal{F}}_{k+1}^{10} + \overline{\underline{\mathcal{F}}_{k+1}^{10}} \text{ with } \underline{\mathcal{F}}_{k+1}^{10} = \{X \in \underline{\mathcal{F}}_k^{10} : [X, \underline{\mathcal{D}}^{01}] \subset \underline{\mathcal{F}}_k^{10} + \underline{\mathcal{D}}^{01}\}, \quad (2.1)$$

and where we set $\underline{\mathcal{F}}_{-1}^{10} = \underline{\mathcal{D}}^{10}$. A CR structure (\mathcal{D}, J) is called *regular* if the vector fields in $\underline{\mathcal{F}}_j$ and $\underline{\mathcal{F}}_j^{10}$ are the sections of corresponding complex distributions $\mathcal{F}_j, \mathcal{F}_j^{10} \subset \mathcal{D}^{\mathbb{C}}$ for any $j \geq -1$. From now on, *any CR manifold will be tacitly assumed to be regular*.

The following is a consequence of definitions.

Lemma 2.1. *If (M, \mathcal{D}, J) is a regular CR manifold, all complex distributions \mathcal{F}_k are J -invariant and real (i.e. equal to their conjugate).*

Furthermore, the class of vector fields in $\mathcal{E} = \text{Re}(\mathcal{F}_0) \subset \mathcal{D}$ is equal to

$$\underline{\mathcal{E}} = \{ X \in \underline{\mathcal{D}} : [X, \underline{\mathcal{D}}] \subset \underline{\mathcal{D}} \} . \quad (2.2)$$

In particular, \mathcal{E} is an involutive subdistribution of \mathcal{D} .

We may now consider the following definition (see e.g. [BER, KZ]).

Definition 2.2. A (regular) CR manifold (M, \mathcal{D}, J) is called *k -nondegenerate* if $\mathcal{F}_j \neq 0$ for all $0 \leq j \leq k-2$ and $\mathcal{F}_{k-1} = 0$. In this case, (M, \mathcal{D}, J) is called *finitely nondegenerate with order of nondegeneracy k* .

If (M, \mathcal{D}, J) is of hypersurface type (i.e. of codimension 1), it is 1-nondegenerate if and only if it is Levi nondegenerate in the usual sense or, equivalently, if and only if \mathcal{D} is a contact distribution.

If (M, \mathcal{D}, J) is of dimension 5 and of hypersurface type, a dimension argument shows that it is finitely nondegenerate if and only if it is either Levi nondegenerate or 2-nondegenerate.

In this paper, we focus on *5-dimensional, CR manifolds of hypersurface type that are 2-nondegenerate*, which we friendly call *girdled CR manifolds*.

By definitions, a girdled CR manifold (M, \mathcal{D}, J) is naturally endowed with the J -invariant, 2-dimensional, involutive subdistribution $\mathcal{E} = \text{Re}(\mathcal{F}_0) \subset \mathcal{D}$, which we call *rib distribution*. Its maximal leaves are called *ribs*: they are complex manifolds of dimension 1 and (M, \mathcal{D}, J) is foliated by such 1-dimensional complex manifolds. However, by finite nondegeneracy, there exist no local CR equivalences between (M, \mathcal{D}, J) and products of the form $\overline{M} \times \mathbb{C}$ for some 3-dimensional CR manifold $(\overline{M}, \overline{\mathcal{D}}, \overline{J})$ (we call them *CR*

straightenings; see also [Fr]). The absence of CR straightenings is the reason why we chose for the word “girdled” for such CR manifolds.

2.2. Levi form and cubic form.

Let (M, \mathcal{D}, J) be a girdled CR manifold with rib distribution $\mathcal{E} \subset \mathcal{D}$ and denote by

$$\mathcal{E}^{10} = \mathcal{D}^{10} \cap \mathcal{E}^{\mathbb{C}}, \quad \mathcal{E}^{01} = \overline{\mathcal{E}^{10}}$$

the holomorphic and antiholomorphic subdistributions in $\mathcal{E}^{\mathbb{C}}$. For any tangent vector $v \in T_x M$, we use the notation $X^{(v)}$ to indicate any vector field, defined on a neighbourhood of x , satisfying the condition

$$X^{(v)}|_x = v.$$

Any such $X^{(v)}$ is said to be a *vector field that extends v around x* .

We call *defining covector at $x \in M$* a 1-form $\vartheta_x \in T_x^* M$ with the property that $\ker \vartheta_x = \mathcal{D}_x$. A 1-form ϑ , defined on an open subset $\mathcal{U} \subset M$, is called *defining 1-form* if ϑ_x is a defining covector for any $x \in \mathcal{U}$.

Lemma 2.3. *Let ϑ be a defining 1-form on a neighbourhood \mathcal{U} of $x \in M$.*

- a) *For any $v, w \in T_x M$ and vector fields $X^{(v)}, X^{(w)} \in \underline{\mathcal{D}}$ that extend v and w around x , we have that*

$$d\vartheta_x(v, Jw) = -\vartheta_x([X^{(v)}, JX^{(w)}]) . \quad (2.1)$$

In particular, $d\vartheta_x(v, Jw)$ depends only on ϑ_x, v and w .

- b) *Given $e \in \mathcal{E}_x^{10}$, $h, h' \in \mathcal{D}_x^{01}$ and $X^{(e)} \in \underline{\mathcal{E}}^{10}$, $X^{(h)}, X^{(h')} \in \underline{\mathcal{D}}^{01}$ that extend e, h and h' , respectively, the corresponding complex number*

$$\vartheta_x([X^{(e)}, X^{(h)}], X^{(h')})$$

depends only on ϑ_x, e, h and h' . Such dependence is linear.

Proof. The equality (2.1) is a consequence of Koszul formula for exterior derivatives. The last claim of (a) follows directly.

For (b), we only need to check that the value of $\vartheta_x([X^{(e)}, X^{(h)}], X^{(h')})$ does not change if one replaces $X^{(e)} \in \underline{\mathcal{E}}^{10}$, $X^{(h)}, X^{(h')} \in \underline{\mathcal{D}}^{01}$ by other extensions $Y^{(e)} \in \underline{\mathcal{E}}^{10}$, $Y^{(h)}, Y^{(h')} \in \underline{\mathcal{D}}^{01}$. They are necessarily of the form

$$Y^{(e)} = \lambda X^{(e)}, \quad Y^{(h)} = \mu \overline{X^{(e)}} + \nu X^{(h)}, \quad Y^{(h')} = \mu' \overline{X^{(e)}} + \nu' X^{(h')}$$

for some \mathbb{C} -valued, smooth functions $\lambda, \mu, \nu, \mu', \nu'$ with

$$\lambda_x = \nu_x = \nu'_x = 1, \quad \mu_x = \mu'_x = 0.$$

Since $[\underline{\mathcal{D}}^{01}, \underline{\mathcal{D}}^{01}] \subset \underline{\mathcal{D}}^{01}$ and $[\underline{\mathcal{E}}^{\mathbb{C}}, \underline{\mathcal{D}}^{\mathbb{C}}] \subset \underline{\mathcal{D}}^{\mathbb{C}}$,

$$[Y^{(e)}, Y^{(h)}] = \lambda \nu [X^{(e)}, X^{(h)}] + X^{(e)}(\nu) X^{(h)} \mod \underline{\mathcal{E}}^{\mathbb{C}}$$

and

$$[[Y^{(e)}, Y^{(h)}], Y^{(h')}] = \lambda \nu \nu' [[X^{(e)}, X^{(h)}], X^{(h')}] \mod \underline{\mathcal{D}}^{\mathbb{C}}.$$

Since ϑ is a defining 1-form and $\lambda_x = \mu_x = \nu_x = 1$

$$\vartheta_x([Y^{(e)}, Y^{(h)}], Y^{(h')}) = \vartheta_x([X^{(e)}, X^{(h)}], X^{(h')}) . \quad \square$$

On the base of the previous lemma, we may consider the following

Definition 2.4. Let $\vartheta_x \in T_x^*M$ be a defining covector at x . We call *Levi form* and *cubic form*, associated with ϑ_x , the linear maps

$$\mathcal{L}^{\vartheta_x} : \mathcal{D}_x \times \mathcal{D}_x \longrightarrow \mathbb{R} , \quad \mathcal{L}^{\vartheta_x}(v, w) = -\vartheta_x([X^{(v)}, JX^{(w)}]) , \quad (2.2)$$

$$\mathcal{H}^{\vartheta_x} : \mathcal{E}_x^{10} \times \mathcal{D}_x^{01} \times \mathcal{D}_x^{01} \longrightarrow \mathbb{C} , \quad \mathcal{H}^{\vartheta_x}(e, h, h') = \vartheta_x([X^{(e)}, X^{(h)}], X^{(h')}) \quad (2.3)$$

for some extensions $X^{(v)}, X^{(w)} \in \underline{\mathcal{D}}$, $X^{(e)} \in \underline{\mathcal{E}}^{10}$, $X^{(h)}, X^{(h')} \in \underline{\mathcal{D}}^{01}$.

3. A MAXIMALLY HOMOGENEOUS MODEL FOR GIRDLED CR MANIFOLDS: THE TUBE OVER THE FUTURE LIGHT CONE

3.1. The tube over the future light cone.

Consider the bilinear form (\cdot, \cdot) and the pseudo-Hermitian form $\langle \cdot, \cdot \rangle$ on \mathbb{C}^5 defined by

$$(t, s) \stackrel{\text{def}}{=} t^T I_{3,2} s , \quad \langle t, s \rangle = (\bar{t}, s) , \quad I_{3,2} = \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & -I_2 \end{array} \right)$$

and the corresponding semi-algebraic subset $M_o \subset \mathbb{C}P^4$ defined by

$$\begin{cases} (t, t) = (t^0)^2 + (t^1)^2 + (t^2)^2 - (t^3)^2 - (t^4)^2 = 0 , \\ \langle t, t \rangle = |t^0|^2 + |t^1|^2 + |t^2|^2 - |t^3|^2 - |t^4|^2 = 0 , \\ \text{Im}(t^3 \bar{t}^4) > 0 . \end{cases} \quad (3.1)$$

One can directly check (see also e.g. [SV]) that M_o is a $\text{SO}_{3,2}^0$ -homogeneous, 5-dimensional CR submanifold of $\mathbb{C}P^4$ ($\text{SO}_{3,2}^0$ = identity component of $\text{SO}_{3,2}$) and contains $\tilde{\mathcal{T}} = M_o \cap \{\text{Im}(t^3 \bar{t}^4) > 0\}$ as open dense subset, which is CR equivalent to the so called *tube over the future light cone* in \mathbb{C}^3 , i.e. the real hypersurface

$$\mathcal{T} = \{ (z^1, z^2, z^3) \in \mathbb{C}^3 : (x^1)^2 + (x^2)^2 - (x^3)^2 = 0 , x^3 > 0 \} . \quad (3.2)$$

In fact, one can directly check that the map $f : \mathbb{C}^3 \longrightarrow \mathbb{C}P^4$ defined by

$$\begin{aligned} f(z^1, z^2, z^3) = & \left[-\frac{i}{2} - \frac{i}{2} ((z^1)^2 + (z^2)^2 - (z^3)^2) : z^1 : \right. \\ & \left. : z^2 : z^3 : -\frac{i}{2} + \frac{i}{2} ((z^1)^2 + (z^2)^2 - (z^3)^2) \right] \end{aligned} \quad (3.3)$$

determines a CR equivalence between \mathcal{T} and $\tilde{\mathcal{T}} \subset M_o$.

It is also known that \mathcal{T} (and M_o as well) is a girdled CR manifold. It is indeed a homogeneous girdled CR manifold with algebra of germs of infinitesimal automorphisms of maximal dimension (see ([FK1])). Indeed the real algebraic variety

$$N = \{ [t] \in \mathbb{C}P^4 : (t, t) = (\bar{t}, t) = 0 \} \subset \mathbb{C}P^4 .$$

is $O_{3,2}$ -invariant and contains exactly two open $SO_{3,2}^0$ -orbits, one of which is M_o . It is known that $\text{Aut}(M_o) = SO_{3,2}^0$, i.e. its CR automorphisms coincide with the transformations determined by the projective actions of the elements in $SO_{3,2}^0$ on M_o (see e.g. [Kl, SV, FK]).

3.2. The graded structure of the Lie algebra of $\text{Aut}(M_o)$.

Consider a system of projective coordinates on \mathbb{CP}^n , in which the bilinear form (\cdot, \cdot) assumes the form

$$(t, s) = t^T \mathcal{I} s \quad \text{with} \quad \mathcal{I} = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (3.4)$$

By means of this new coordinates, $\mathfrak{so}_{3,2}(\mathbb{R})$ can be identified with the Lie algebra of real matrices such that $A^T \mathcal{I} + \mathcal{I} A = 0$, i.e., of the form

$$A = \left(\begin{array}{cc|cc|cc} a_1 & a_2 & a_5 & a_7 & 0 & \\ a_3 & a_4 & a_6 & 0 & -a_7 & \\ \hline a_8 & a_9 & 0 & -a_6 & -a_5 & \\ a_{10} & 0 & -a_9 & -a_4 & -a_2 & \\ \hline 0 & -a_{10} & -a_8 & -a_3 & -a_1 & \end{array} \right), \quad \text{for some } a_i \in \mathbb{R}.$$

In particular, it admits a basis \mathcal{B} , given by the matrices

$$\begin{aligned} e^{-2} &= \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \end{array} \right), \quad e_1^{-1} = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 \end{array} \right), \\ e_2^{-1} &= \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad e_1^0 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right), \\ e_2^0 &= \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right), \quad E_1^0 = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right), \\ E_2^0 &= \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right), \quad E_1^1 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \\ E_2^1 &= \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad E^2 = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (3.5)$$

For the discussions of the next sections, it is quite useful to have all Lie brackets between elements of the basis \mathcal{B} explicitly written down. Moreover,

in place of the elements E_j^i and e_j^{-i} , for $i = 0, 1$ and $j = 1, 2$, it is convenient to consider the elements in $\mathfrak{so}_{3,2}(\mathbb{R})^{\mathbb{C}}$

$$\begin{aligned} E^{\ell(10)} &= \frac{1}{2} (E_1^\ell - iE_2^\ell) , & e^{-\ell(10)} &= \frac{1}{2} (e_1^{-\ell} - ie_2^{-\ell}) , \\ E^{\ell(01)} &= \overline{E^{\ell(10)}} , & e^{-\ell(01)} &= \overline{e^{-\ell(10)}} , \end{aligned} \quad \ell = 0, 1 , \quad (3.6)$$

and evaluate the Lie brackets between such vectors and between them and the real vectors E^2 , e^{-2} or E_1^0 . Here is the list of such Lie brackets.

	E^2	$E^{1(10)}$	$E^{1(01)}$	$E^{0(10)}$	$E^{0(01)}$	$e^{0(10)}$	$e^{0(01)}$	$e^{-1(10)}$	$e^{-1(01)}$	e^{-2}
$\text{ad}_{E_1^0}$	$2E^2$	$E^{1(10)}$	$E^{1(01)}$	0	0	0	0	$-e^{-1(10)}$	$-e^{-1(01)}$	$-2e^{-2}$
ad_{E^2}	0	0	0	$-E^2$	$-E^2$	0	0	$iE^{1(10)}$	$-iE^{1(01)}$	$\frac{E^{0(10)} + E^{0(01)}}{+}$
$\text{ad}_{E^{1(10)}}$	0	0	$-\frac{i}{2}E^2$	0	$-E^{1(10)}$	0	$-E^{1(01)}$	$\frac{1}{2}e^{0(10)}$	$\frac{1}{2}E^{0(01)}$	$ie^{-1(10)}$
$\text{ad}_{E^{1(01)}}$	0	$\frac{i}{2}E^2$	0	$-E^{1(01)}$	0	$-E^{1(10)}$	0	$\frac{1}{2}E^{0(10)}$	$\frac{1}{2}e^{0(01)}$	$-ie^{-1(01)}$
$\text{ad}_{E^{0(10)}}$	E^2	0	$E^{1(01)}$	0	0	$-e^{0(10)}$	$e^{0(01)}$	$-e^{-1(10)}$	0	$-e^{-2}$
$\text{ad}_{E^{0(01)}}$	E^2	$E^{1(10)}$	0	0	0	$e^{0(10)}$	$-e^{0(01)}$	0	$-e^{-1(01)}$	$-e^{-2}$
$\text{ad}_{e^{0(10)}}$	0	0	$E^{1(10)}$	$e^{0(10)}$	$-e^{0(10)}$	0	$\frac{-E^{0(10)} + E^{0(01)}}{+}$	0	$-e^{-1(10)}$	0
$\text{ad}_{e^{0(01)}}$	0	$E^{1(01)}$	0	$-e^{0(01)}$	$e^{0(01)}$	$\frac{E^{0(10)} - E^{0(01)}}{-}$	0	$-e^{-1(01)}$	0	0
$\text{ad}_{e^{-1(10)}}$	$-iE^{1(10)}$	$-\frac{1}{2}e^{0(10)}$	$-\frac{1}{2}E^{0(10)}$	$e^{-1(10)}$	0	0	$e^{-1(01)}$	0	$\frac{i}{2}e^{-2}$	0
$\text{ad}_{e^{-1(01)}}$	$iE^{1(01)}$	$-\frac{1}{2}E^{0(01)}$	$-\frac{1}{2}e^{0(01)}$	0	$e^{-1(01)}$	$e^{-1(10)}$	0	$-\frac{i}{2}e^{-2}$	0	0
$\text{ad}_{e^{-2}}$	$\frac{-E^{0(10)} - E^{0(01)}}{-}$	$-ie^{-1(10)}$	$ie^{-1(01)}$	e^{-2}	e^{-2}	0	0	0	0	0

Table 1

The brackets between all the elements of \mathcal{B} can be directly recovered from Table 1 recalling that

$$\begin{aligned} E_1^\ell &= E^{\ell(10)} + E^{\ell(01)} , & E_2^\ell &= i(E^{\ell(10)} - E^{\ell(01)}) , \\ e_1^{-\ell} &= e^{-\ell(10)} + e^{-\ell(01)} , & e_2^{-\ell} &= i(e^{-\ell(10)} - e^{-\ell(01)}) , \end{aligned} \quad \text{for } \ell = 0, 1 . \quad (3.7)$$

Notice that $\mathfrak{so}_{3,2}$ has a natural graded Lie algebra structure

$$\mathfrak{so}_{3,2} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 ,$$

given by the eigenspaces of the adjoint action of the *grading element* E_1^0

$$\mathfrak{g}^{-2} = \langle e^{-2} \rangle , \quad \mathfrak{g}^{-1} = \langle e_1^{-1}, e_2^{-1} \rangle , \quad \mathfrak{g}^0 = \langle e_1^0, e_2^0, E_1^0, E_2^0 \rangle ,$$

$$\mathfrak{g}^1 = \langle E_1^1, E_2^1 \rangle , \quad \mathfrak{g}^2 = \langle E^2 \rangle .$$

3.3. The Levi form and the cubic form of M_o .

Consider the point $x_o = [1 : i : 0 : 0 : 0]$ of M_o (we are using the coordinates defined by (3.4)). The isotropy subalgebra $\mathfrak{h} = \text{Lie}(\text{Aut}_{x_o}(M_o))$ consists of the matrices $A \in \mathfrak{so}_{3,2}(\mathbb{R})$ such that

$$A \cdot (1, i, 0, 0, 0) = \lambda(1, i, 0, 0, 0) \quad \text{for some } 0 \neq \lambda \in \mathbb{C}.$$

From this, one can directly check that

$$\mathfrak{h} = \langle E_1^0, E_2^0, E_1^1, E_2^1, E^2 \rangle = \mathfrak{h}^0 + \mathfrak{g}^1 + \mathfrak{g}^2,$$

where $\mathfrak{h}^0 \subset \mathfrak{g}^0$ is the subspace $\mathfrak{h}^0 = \langle E_1^0, E_2^0 \rangle$. If we denote by $\mathfrak{m}^0 \subset \mathfrak{g}^0$ the subspace $\mathfrak{m}^0 = \langle e_1^0, e_2^0 \rangle$, we see that $\mathfrak{m} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{m}^0$ is complementary to \mathfrak{h} in \mathfrak{g} and that the linear map

$$\iota : \mathfrak{m} \longrightarrow T_{x_o}M, \quad X \in \mathfrak{m} \longmapsto \underline{X} \stackrel{\text{def}}{=} \widehat{X}|_{x_o} \in T_{x_o}M_o$$

where each \widehat{X} is the vector field on M_o with flow equal to $\exp(tX) \cdot : M_o \longrightarrow M_o$, $t \in \mathbb{R}$, is an isomorphism. We use ι to identify \mathfrak{m} with $T_{x_o}M$ and, for any $v \in \mathfrak{m}$, we set $\underline{v} = \iota(v)$.

In this way, the basis $\mathcal{B}^{\mathfrak{m}} = (e^{-2}, e_i^{-1}, e_i^0)$ of \mathfrak{m} can be identified with the basis $\underline{\mathcal{B}}^{\mathfrak{m}}$ of $T_{x_o}M_o$ formed by the vectors

$$\begin{aligned} \underline{e}^{-2} &= \text{Re} \left(\frac{\partial}{\partial z^3} \Big|_{x_o} - i \frac{\partial}{\partial z^4} \Big|_{x_o} \right), \\ \underline{e}_1^{-1} &= \text{Re} \left(\frac{\partial}{\partial z^2} \Big|_{x_o} \right), \quad \underline{e}_2^{-1} = \text{Re} \left(i \frac{\partial}{\partial z^2} \Big|_{x_o} \right), \\ \underline{e}_1^0 &= \text{Re} \left(\frac{\partial}{\partial z^0} \Big|_{x_o} - i \frac{\partial}{\partial z^1} \Big|_{x_o} \right), \quad \underline{e}_2^0 = \text{Re} \left(i \frac{\partial}{\partial z^0} \Big|_{x_o} + \frac{\partial}{\partial z^1} \Big|_{x_o} \right). \end{aligned}$$

From this and with the help of Table 1, one can check that the invariant CR distribution \mathcal{D} and the rib distribution \mathcal{E} are given by

$$\mathcal{D}_{x_o} = \langle \underline{e}_1^{-1}, \underline{e}_2^{-1}, \underline{e}_1^0, \underline{e}_2^0 \rangle = \iota(\mathfrak{g}^{-1} + \mathfrak{m}^0), \quad \mathcal{E}_{x_o} = \langle \underline{e}_1^0, \underline{e}_2^0 \rangle = \iota(\mathfrak{m}^0), \quad (3.8)$$

and the invariant complex structure J on \mathcal{D} is such that

$$J(\underline{e}_1^{-1}) = \underline{e}_2^{-1}, \quad J(\underline{e}_1^0) = \underline{e}_2^0.$$

Moreover, if we denote by $\underline{\mathcal{B}}^{\mathfrak{m}*}$ the dual basis of $\underline{\mathcal{B}}^{\mathfrak{m}}$, direct computations show that

$$\vartheta = (\underline{e}^{-2})^* \quad (= \text{the 1-form dual to } \underline{e}^{-2})$$

is a defining covector for \mathcal{D} , with corresponding Levi form and cubic form equal to

$$\mathcal{L}^\vartheta(\underline{e}^{-1(10)}, \underline{e}^{-1(01)}) = -\frac{1}{2}, \quad \mathcal{H}^\vartheta(\underline{e}^{0(10)}, \underline{e}^{-1(01)}, \underline{e}^{-1(01)}) = -\frac{i}{2} \quad (3.9)$$

(here, $\underline{e}^{-\ell(10)} = \frac{1}{2}(\underline{e}_1^{-\ell} - iJ\underline{e}_1^{-\ell})$ and $\underline{e}^{-\ell(01)} = \overline{e^{-\ell(10)}}$). In fact, one may consider the connected subgroup $\mathbb{M} \subset \mathrm{SO}_{3,2}^0 = \mathrm{Aut}(M_o)$ with $\mathrm{Lie}(\mathbb{M}) = \mathfrak{m}$ and observe that the restriction $\pi|_{\mathbb{M}}$ of the natural projection

$$\pi : \mathrm{SO}_{3,2}^0 = \mathrm{Aut}(M_o) \longrightarrow M_o \simeq \mathrm{Aut}(M_o) / \mathrm{Aut}_{x_o}(M_o)$$

determines a (local) diffeomorphism $\pi|_{\mathbb{M}} : \mathbb{M} \longrightarrow M_o$. By $\mathrm{SO}_{3,2}^0$ -invariance and (3.8), the pull-back (\mathbb{D}, \mathbb{J}) on \mathbb{M} of the CR-structure (\mathcal{D}, J) , is formed by the left-invariant distribution \mathbb{D} , spanned by $e_i^{-1}, e_j^0, i, j = 1, 2$, and the complex structure \mathbb{J} such that $\mathbb{J}(e_1^i) = e_2^i$ for any $i = -1, 0$. From this, the Levi form and the cubic form of $(\mathbb{M}, \mathbb{D}, \mathbb{J})$ (and of (M, \mathcal{D}, J)) can be directly computed using Table 1.

3.4. The CR structure of M_o and the complex structure $J|_{\mathfrak{m}^{-1} + \mathfrak{m}^0}$.

The graded subspaces $\mathfrak{g}^{-2}, \mathfrak{g}^{-1}, \mathfrak{g}^1, \mathfrak{g}^2 \subset \mathfrak{so}_{3,2}$ will be often indicated with the symbols $\mathfrak{m}^{-2}, \mathfrak{m}^{-1}, \mathfrak{h}^1, \mathfrak{h}^2$, respectively, so that the graded decomposition of \mathfrak{m} and \mathfrak{h} are

$$\mathfrak{m} = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0 \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2.$$

We also denote by J the grade preserving complex structure on the subalgebra $\mathfrak{m}^{-1} + \mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1$, defined by

$$J(e_1^{-1}) = J(e_2^{-1}), \quad J(e_1^0) = J(e_2^0), \quad J(E_1^0) = E_2^0, \quad J(E_1^1) = E_2^1. \quad (3.10)$$

Notice that:

- the vectors $e^{-\ell(10)}, E^{\ell(10)}$ and $e^{-\ell(01)}, E^{\ell(01)}$, introduced in (3.6), are the J -holomorphic and J -antiholomorphic parts of the elements $e_1^{-\ell}, E_1^{\ell}$;
- through the isomorphism $\iota : \mathfrak{m} \longrightarrow T_{x_o}M_o$, the restriction $J|_{\mathfrak{m}^{-1} + \mathfrak{m}^0}$ corresponds to the complex structure J of \mathcal{D}_{x_o} and completely determines the invariant CR structure of M_o ;
- using the isomorphism ι , the vectors $e^{-\ell(10)}, \ell = 0, 1$, can be considered as a basis for $\mathcal{D}_{x_o}^{10} \subset T_{x_o}^{\mathbb{C}}M_o$.

4. FILTERED VECTOR SPACES MODELLED ON $\mathfrak{so}_{3,2}(\mathbb{R})$

4.1. Filtrations with an additional semitone.

In the following sections, we have to study the geometric structures on manifolds modelled on the Lie algebra $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R})$. In order to do this, we want to establish a few properties of filtrations of vector spaces that are similar to some special filtrations of $\mathfrak{so}_{3,2}(\mathbb{R})$.

Consider a finite-dimensional vector space V endowed with a filtration of the following kind

$$\mathcal{F} : V = V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset \cdots \supset V_k \supset \{0\}. \quad (4.1)$$

If V_0 has its own filtration with an extra term $V_{(0|0)}$ between V_0 and V_1 , i.e.

$$\underbrace{V_{(0|-1)}}_{=V_0} \supsetneq V_{(0|0)} \supset \underbrace{V_{(0|1)}}_{=V_1} \supset \dots \supset \underbrace{V_{(0|k)}}_{=V_k} \supset \{0\} , \quad (4.2)$$

we may merge such two filtrations and obtain a new one, namely

$$\mathcal{F}_* : V = V_{-2} \supset V_{-1} \supset V_0 \supsetneq V_{(0|0)} \supset V_1 \supset \dots \supset V_k \supset \{0\} . \quad (4.3)$$

Such new filtration is called *filtration with one additional semitone*. The main example to have in mind is given by the two filtrations of $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R})$

$$\begin{aligned} \mathcal{F} : \underbrace{\mathfrak{m} + \mathfrak{h}}_{V_{-2}} \supset \underbrace{\mathfrak{m}^{-1} + (\mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2)}_{V_{-1}} \supset \underbrace{\mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2}_{V_0} \supset \\ \supset \underbrace{\mathfrak{h}^1 + \mathfrak{h}^2}_{V_1} \supset \underbrace{\mathfrak{h}^2}_{V_2} \supset \{0\} , \end{aligned} \quad (4.4)$$

$$\begin{aligned} \mathcal{F}_* : \underbrace{\mathfrak{m} + \mathfrak{h}}_{V_{-2}} \supset \underbrace{\mathfrak{m}^{-1} + (\mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2)}_{V_{-1}} \supset \underbrace{\mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2}_{V_0=V_{(0|-1)}} \supset \\ \supset \underbrace{\mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2}_{V_{(0|0)}} \supset \underbrace{\mathfrak{h}^1 + \mathfrak{h}^2}_{V_{(0|1)}=V_1} \supset \underbrace{\mathfrak{h}^2}_{V_{(0|2)}=V_2} \supset \{0\} . \end{aligned} \quad (4.5)$$

In the following, given filtrations (4.1) and (4.3), we denote by $\mathrm{GL}(V, \mathcal{F})$, $\mathrm{GL}(V, \mathcal{F}_*)$ and $\mathfrak{gl}(V, \mathcal{F})$, $\mathfrak{gl}(V, \mathcal{F}_*)$ the Lie groups and Lie algebras of filtration preserving automorphisms of (V, \mathcal{F}) and (V, \mathcal{F}_*) , respectively. Notice that $\mathfrak{gl}(V, \mathcal{F})$ and $\mathfrak{gl}(V, \mathcal{F}_*)$ are naturally endowed with structures of filtered Lie algebras, with filtrations determined by the subspaces

$$\begin{aligned} \mathfrak{gl}_i(V) &= \{ A \in \mathfrak{gl}(V, \mathcal{F}) : A(V_k) \subset V_{k+i} \text{ for any } k \geq -2 \} , \\ \mathfrak{gl}_{i*}(V) &= \{ A \in \mathfrak{gl}(V, \mathcal{F}_*) : A(V_{-2}) \subset V_{-2+i} , A(V_{-1}) \subset V_{-1+i} , \\ &\quad A(V_{(0|k)}) \subset V_{(0|k+i)} \text{ for } k \geq -1 \} , \quad i \geq 0 . \end{aligned}$$

We denote by $\mathrm{GL}_i(V)$ and $\mathrm{GL}_{i*}(V)$ the corresponding connected subgroups of $\mathrm{GL}(V, \mathcal{F})$ and $\mathrm{GL}(V, \mathcal{F}_*)$, respectively.

Let $W = \mathfrak{gr}(V, \mathcal{F})$ and $W_* = \mathfrak{gr}(V, \mathcal{F}_*)$ the graded vector spaces of (V, \mathcal{F}) and (V, \mathcal{F}_*) , i.e.

$$W = W^{-2} + W^{-1} + W^0 + W^1 + \dots + W^k , \quad (4.6)$$

$$W_* = W^{-2} + W^{-1} + W^{(0|-1)} + W^{(0|0)} + W^1 + \dots + W^k \quad (4.7)$$

with $W^i = V_i/V_{i+1}$ and $W^{(0|k)} = V_{(0|k)}/V_{(0|k+1)}$. The corresponding natural projections are denoted by

$$\pi^i : V_i \longrightarrow W^i = V_i/V_{i+1} , \quad \pi^{(0|j)} : V_j \longrightarrow W^{(0|j)} = V_{(0|j)}/V_{(0|j+1)} .$$

Note that the graded vector spaces W , W_* are naturally endowed with filtrations, which we denote \mathcal{F} and \mathcal{F}_* , respectively (the filtration \mathcal{F} is $\mathcal{F} = \{W_j = \sum_{i \geq j} W^i\}$; the filtration \mathcal{F}_* is defined analogously), so that also the Lie groups $\mathrm{GL}_i(W)$ and $\mathrm{GL}_{i*}(W_*)$ are well defined.

4.2. Partial complex structures on filtered vector spaces.

From §3.4, we know that there exist two complex structures on the subspaces $\mathfrak{m}^{-1}, \mathfrak{m}^0$ of $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R})$, which are algebraic counterparts of the CR structure of M_o . Motivated by this, we consider the following

Definition 4.1. A *partial complex structure* on (V, \mathcal{F}_*) is a filtration preserving linear map $J : V_{-1} \longrightarrow V_{-1}$ such that, for any $X \in V_{-1}$,

$$J^2 = -\text{Id}_{V_{-1}} \mod V_{(0|0)} . \quad (4.8)$$

Two partial complex structures J, J' are called *equivalent* if

$$J - J' = 0 \mod V_{(0|0)} .$$

A partial complex structure J induces on $W^{-1} + W^{(0|-1)}$ the complex structure defined by

$$J(X_{\text{mod } V_0}) = J(X)_{\text{mod } V_0} , \quad J(X_{\text{mod } V_{(0|0)}}) = J(X)_{\text{mod } V_{(0|0)}} .$$

Note that

- equivalent partial structures induce the same complex structure on $W^{-1} + W^{(0|-1)}$;
- if J is extended to an endomorphism of W_{-1} , it is a partial complex structure on $W_* = \mathfrak{gr}(V, \mathcal{F}_*)$; any two such extensions are equivalent.

We use the notation

$$\text{GL}(V, \mathcal{F}_*, J) = \{ A \in \text{GL}(V, \mathcal{F}_*) : J \circ A|_{V_{-1}} = A \circ J|_{V_{-1}} \mod V_{(0|0)} \} ,$$

$$\text{GL}_{i*}(V, J) = \text{GL}(V, \mathcal{F}_*, J) \cap \text{GL}_{i*}(V) ,$$

$$\text{GL}_i(V, J) = \text{GL}(V, \mathcal{F}_*, J) \cap \text{GL}_i(V) ,$$

$$\begin{aligned} \text{GL}^{\text{gr}}(W, J) &= \{ A \in \text{GL}(W) : A \text{ maps graded subspaces into graded} \\ &\quad \text{subspaces and } J \circ A|_{W^{-1} + W^{(0|-1)}} = A \circ J|_{W^{-1} + W^{(0|-1)}} \mod W_{(0|0)} \} , \end{aligned}$$

$$\text{GL}_i^{\text{gr}}(W, J) = \text{GL}^{\text{gr}}(W, J) \cap \text{GL}_i(W, J) .$$

Lie algebras of such groups are denoted by the corresponding gothic letters.

Remark 4.2. By definitions, for any $i \geq 2$, the groups $\text{GL}_i(V, J)$ and $\text{GL}_{i*}(V, J)$ coincide with the groups $\text{GL}_i(V)$ and $\text{GL}_{i*}(V)$, respectively.

Given two vector spaces V, V' , both with filtrations (4.1), (4.3) and partial complex structures J, J' , respectively, we call *filtered (J, J') -isomorphism* any filtration preserving isomorphism $u : V \longrightarrow V'$ such that

$$u \circ J = J' \circ u|_{V_{-1}} \mod V'_{(0|0)} .$$

Similarly, given two graded vector spaces W_* , W'_* , with gradations (4.7) and partial complex structures J , J' , we call *graded (J, J') -isomorphism* a grading preserving isomorphism $\varphi : W \rightarrow W'$ such that

$$\varphi \circ J|_{W^{-1}+W^{(0|-1)}} = J' \circ \varphi|_{W^{-1}+W^{(0|-1)}} .$$

Consider now the Lie algebra $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R})$, endowed with the filtrations (4.1) and (4.1) and the complex structure (3.10) on $\mathfrak{m}^{-1} + \mathfrak{m}^0 + \mathfrak{h}^0 + \mathfrak{h}^1$. Any linear extension of J onto the entire subspace $\mathfrak{m}_{(0|-2)} = \mathfrak{m}^{-1} + \mathfrak{m}^0 + \mathfrak{h}$ is a partial complex structure of \mathfrak{g} and two such extensions are equivalent. This property is indeed the main motivation for the definitions considered in this section.

We conclude observing that, on the subalgebra $\mathfrak{m} = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0 \subset \mathfrak{so}_{3,2}$, one has that $\mathcal{F}^* = \mathcal{F}$ so that $\mathrm{GL}_{1*}(\mathfrak{m}, J) = \mathrm{GL}_1(\mathfrak{m}, J)$.

4.3. Adapted frames on spaces with partial complex structure.

In the next lemma, V is a vector space with filtrations (4.1), (4.3) and with a partial complex structure J . The associated graded vector spaces are $W = \mathfrak{gr}(V, \mathcal{F})$, $W_* = \mathfrak{gr}(V, \mathcal{F}_*)$ and we denote by $\tilde{\mathfrak{m}}$ a (modelling) graded vector space

$$\tilde{\mathfrak{m}} = \underset{=\tilde{\mathfrak{m}}^{(0|-3)}}{\tilde{\mathfrak{m}}^{-2}} + \underset{=\tilde{\mathfrak{m}}^{(0|-2)}}{\tilde{\mathfrak{m}}^{-1}} + \tilde{\mathfrak{m}}^{(0|-1)} + \tilde{\mathfrak{m}}^{(0|0)} + \underset{=\tilde{\mathfrak{m}}^{(0|1)}}{\tilde{\mathfrak{m}}^1} + \dots + \underset{=\tilde{\mathfrak{m}}^{(0|k)}}{\tilde{\mathfrak{m}}^k} ,$$

which is isomorphic to W_* as graded vector space. The subspace $\tilde{\mathfrak{m}}^{-1} + \tilde{\mathfrak{m}}^{(0|-1)}$ is assumed endowed with a graded complex structure \tilde{J} , which we consider extended to some graded endomorphism of $\tilde{\mathfrak{m}}_{-1} = \sum_{j \geq -1} \tilde{\mathfrak{m}}^j$, so that it can be considered as a partial complex structure on $(\tilde{\mathfrak{m}}, \tilde{\mathcal{F}}_*)$, with filtration $\tilde{\mathcal{F}}_* = \{ \tilde{\mathfrak{m}}_{(0|j)} = \sum_{s \geq j} \tilde{\mathfrak{m}}^{(0|s)} \}$.

Lemma 4.3. *For any graded (\tilde{J}, J) -isomorphism $u : \tilde{\mathfrak{m}} \rightarrow W$, there exists a filtered (\tilde{J}, J) -isomorphism $u_{\sharp} : \tilde{\mathfrak{m}} \rightarrow V$ satisfying the condition*

$$u_{\sharp}(\tilde{\mathfrak{m}}_{(0|j)}) \subset V_{(0|j)} \quad \text{and} \quad u|_{\tilde{\mathfrak{m}}_{(0|j)}} = \pi^{(0|j)} \circ u_{\sharp}|_{\tilde{\mathfrak{m}}_{(0|j)}} \quad (4.9)$$

for any $j \geq -3$. Any two such isomorphisms u_{\sharp} , $u'_{\sharp} : \tilde{\mathfrak{m}} \rightarrow V$ are related by an element $\mathrm{GL}_{1*}(\tilde{\mathfrak{m}}, \tilde{J})$, i.e.,

$$u'_{\sharp} = u_{\sharp} \circ A \quad \text{for some } A \in \mathrm{GL}_{1*}(\tilde{\mathfrak{m}}, \tilde{J}) .$$

Proof. For each $i < k$, consider a subspace $H^{(0|i)}$ of $V_{(0|i)}$ that is complementary to $V_{(0|i+1)}$. Assume also that the spaces $H^{(0|-2)}$ and $H^{(0|-1)}$ are J -invariant, modulo elements in $V_{(0|0)}$. The map $\pi = \sum_{j \geq -3} \pi^{(0|j)}$ determines a graded isomorphism between the graded vector space

$$V = H^{(0|-3)} + \dots + H^{(0|k-1)} + V_{(0|k)}$$

and $W = \mathfrak{gr}(V)$ and $u_{\sharp} = \pi^{-1} \circ u$ is the desired isomorphism. The last claim is immediate. \square

The proof of Lemma 4.3 shows that, for a fixed graded (\tilde{J}, J) -isomorphism $u : \tilde{\mathfrak{m}} \rightarrow W$ there is a natural one to one correspondence between isomorphisms $u_{\sharp} : \tilde{\mathfrak{m}} \rightarrow V$ satisfying (4.9) and ordered sequences

$$H = (\underset{=H^{(0|-3)}}{H^{-2}}, \underset{=H^{(0|-2)}}{H^{-1}}, H^{(0|-1)}, H^{(0|0)}, \underset{=H^{(0|1)}}{H^1}, \dots, \underset{=H^{(0|k-1)}}{H^{k-1}}) , \quad (4.10)$$

formed by subspaces $H^{(0|i)} \subset V_{(0|i)}$ that are complementary to $V_{(0|i+1)}$ and such that $H^{(0|-2)}$ and $H^{(0|-1)}$ are J -invariant modulo $V_{(0|0)}$.

Any such sequence (4.10) is called *sequence of \mathcal{F}_* -horizontal subspaces of V* or, shortly, *s.h.s. of V* . For a fixed graded (\tilde{J}, J) -isomorphism $u : \tilde{\mathfrak{m}} \rightarrow W$, the corresponding the (\tilde{J}, J) -isomorphism, determined by an s.h.s. H , is called *linear frame of V associated with u and H* and denoted by

$$u_H : \tilde{\mathfrak{m}} \rightarrow V$$

or simply $\hat{H} : \tilde{\mathfrak{m}} \rightarrow V$, in case u is considered as known and fixed. The linear frames constructed in this way are called *adapted to the filtration and the partial complex structure of V* , or *adapted* for short.

5. THE TANAKA STRUCTURE OF A GIRDLED CR MANIFOLD

5.1. Adapted frames of a girdled CR manifold.

From now on, (M, \mathcal{D}, J) is a girdled CR manifold with rib distribution \mathcal{E} . Any tangent space $T_x M$ of M is naturally endowed with a filtration of the form (4.1), namely

$$T_x M \supset \underset{=V_{-2}}{\mathcal{D}_x} \supset \underset{=V_{-1}}{\mathcal{E}_x} \supset \{0\} , \quad (5.1)$$

with graded vector space $\mathfrak{gr}(T_x M, \mathcal{F})$ isomorphic to the subalgebra $\mathfrak{m} = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0$ of $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R})$. Considering the filtrations (4.3) with $V_{(0|0)} = \{0\}$ (and hence with $\mathcal{F} = \mathcal{F}_*$) and the complex structures $J_x : \mathcal{D}_x \rightarrow \mathcal{D}_x$ of the CR structure, we see that the $T_x M$'s are naturally endowed with partial complex structures. We may therefore consider the class $\mathcal{F}r_0(M)$ of (J, J) -isomorphisms

$$u_{\sharp} : \mathfrak{m} = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0 \rightarrow T_x M , \quad x \in M ,$$

adapted to such filtrations and partial complex structures, i.e., given by some graded (J, J) -isomorphism $u : \mathfrak{m} \rightarrow \mathfrak{gr}(T_x M)$ and a s.h.s. $H \subset T_x M$.

If $\pi_0 : \mathcal{F}r_0(M) \rightarrow M$ is the natural projection, from Lemma 4.3 it follows that $\pi_0 : \mathcal{F}r_0(M) \rightarrow M$ is a principal bundle with structure group

$$G_{\sharp}^0 = \mathrm{GL}_0^{\mathrm{gr}}(\mathfrak{m}, J) \ltimes \mathrm{GL}_1(\mathfrak{m}, J) = \mathrm{GL}_0^{\mathrm{gr}}(\mathfrak{m}, J) \ltimes \mathrm{GL}_{1*}(\mathfrak{m}, J) .$$

Note that any adapted linear frame $u_{\sharp} : \mathfrak{m} \rightarrow T_x M$ is uniquely determined by the corresponding frame $(f^{-2}, f_1^{-1}, f_2^{-1}, f_1^0, f_2^0)$ of $T_x M$

$$f^{-2} = u_{\sharp}(e^{-2}) , \quad f_j^{-1} = u_{\sharp}(e_j^{-1}) , \quad f_j^0 = u_{\sharp}(e_j^0) ,$$

or by its dual coframe $(f^{-2*}, f_1^{-1*}, f_2^{-1*}, f_1^{0*}, f_2^{0*}) \subset T_x^* M$, for which

$$\mathcal{D}_x = \ker f^{-2*} , \quad \mathcal{E}_x = \ker f^{-2*} \cap \ker f_1^{-1*} \cap \ker f_2^{-1*} .$$

In the following, for any $u \in \mathcal{F}r_0(M)$, we denote by \mathcal{L}^u and \mathcal{H}^u the Levi form and the cubic form determined by the defining covector f^{-2*} .

5.2. Strongly adapted frames of (M, \mathcal{D}, J) .

Definition 5.1. A *strongly adapted frame* of $T_x M$ is an adapted frame $u_{\sharp} = (f^{-2}, f_1^{-1}, \dots) : \mathfrak{m} \longrightarrow T_x M$ such that (compare with (3.9))

$$\mathcal{L}^u(f^{-1(10)}, \overline{f^{-1(10)}}) = -\frac{1}{2}, \quad \mathcal{H}^u(f^{0(10)}, \overline{f^{-1(10)}}, \overline{f^{-1(10)}}) = -\frac{i}{2}, \quad (5.2)$$

where we set $f^{0(10)} = u(e^{0(10)})$, $f^{-1(10)} = u(e^{-1(10)})$.

The following lemma is a direct consequence of definitions.

Lemma 5.2. *The subset $P_{\sharp}^0 \subset \mathcal{F}r_0(M)$ of strongly adapted frames is a reduction of $\mathcal{F}r_0(M)$ with structure group*

$$H_{\sharp}^0 = H^0 \ltimes \mathrm{GL}_1(\mathfrak{m}, J),$$

where $H^0 \subset \mathrm{GL}_0^{\mathrm{gr}}(\mathfrak{m}, J)$ is the subgroup of maps A such that

$$[A(X), \overline{A(Y)}] = A([X, \overline{Y}]) \quad , \quad [[A(Z), \overline{A(X)}], \overline{A(Y)}] = A([[Z, \overline{X}], \overline{Y}]) \quad (5.3)$$

for any $X, Y \in \mathfrak{m}^{-1(10)}$, $Z \in \mathfrak{m}^{0(10)}$.

Since $\mathrm{GL}_1(\mathfrak{m}, J)$ is normal in H_{\sharp}^0 , we may consider the quotient bundle

$$\pi^0 : P^0 = P_{\sharp}^0 / \mathrm{GL}_1(\mathfrak{m}, J) \longrightarrow M,$$

which is a principal H^0 -bundle. Motivated by Tanaka's theory ([Ta1, Ta2, Ta3, AS]), we call it *Tanaka structure of (M, \mathcal{D}, J)* .

Remark 5.3. We recall that any adapted frame $u_{\sharp} : \mathfrak{m} \longrightarrow T_x M$ is uniquely determined by the induced isomorphism $u : \mathfrak{m} \longrightarrow \mathfrak{gr}(T_x M, \mathcal{F})$ and a s.h.s. $H = (H^{-2}, H^{-1})$ of $T_x M$. Note that two adapted frames u_{\sharp}, u'_{\sharp} are in the same equivalence class $[u_{\sharp}] \in P^0|_x$ if and only if they determine the same graded isomorphism u . Hence, P^0 can be also defined as the *bundle of J -preserving, graded isomorphisms $u : \mathfrak{m} \longrightarrow \mathfrak{gr}(T_x M, \mathcal{F})$, determined by frames satisfying (5.2)*.

Lemma 5.4. *The Lie algebra of H^0 is 2-dimensional and equal to*

$$\mathrm{Lie}(H^0) = \mathrm{ad}(\mathfrak{h}^0)|_{\mathfrak{m}},$$

where $\mathrm{ad}(\mathfrak{h}^0)|_{\mathfrak{m}} \subset \mathfrak{gl}_0(\mathfrak{m}, J)$ denotes the subalgebra of the restrictions $\mathrm{ad}(X)|_{\mathfrak{m}}$, $X \in \mathfrak{h}^0 \subset \mathfrak{so}_{3,2}(\mathbb{R})$. In particular, $\mathrm{Lie}(H^0) \simeq \mathfrak{h}^0$ and it is naturally endowed with the complex structure $J|_{\mathfrak{h}^0}$, defined in (3.10). Moreover, H^0 is isomorphic with the connected subgroup of $H \subset \mathrm{SO}_{3,2}^0$ with $\mathrm{Lie}(H^0) = \mathfrak{h}^0$.

Proof. A map $B \in \mathfrak{gl}_0^{\mathrm{gr}}(\mathfrak{m}, J)$ is of the form

$$B(e^{-2}) = \tau e^{-2}, \quad B(e^{-1(10)}) = \lambda e^{-1(10)} = \overline{B(e^{-1(01)})}, \quad B(e^{0(10)}) = \mu e^{0(10)}$$

for some $\tau \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$. On the other hand, B is in $Lie(H^0)$ if and only if

$$[B(e^{-1(10)}), e^{-1(01)}] + [e^{-1(10)}, B(e^{-1(01)})] = B([e^{-1(10)}, e^{-1(01)}]) = \frac{i}{2}B(e^{-2}), \quad (5.4)$$

$$\begin{aligned} & [[B(e^{0(10)}), e^{-1(01)}], e^{-1(01)}] + [[e^{0(10)}, B(e^{-1(01)})], e^{-1(01)}] + \\ & + [[e^{0(10)}, e^{-1(01)}], B(e^{-1(01)})] = B([e^{0(10)}e^{-1(01)}], e^{-1(01)}) = \frac{-i}{2}B(e^{-2}). \end{aligned} \quad (5.5)$$

These conditions are equivalent to $\tau = 2 \operatorname{Re}(\lambda)$ and $\mu = \tau - 2\bar{\lambda} = 2i \operatorname{Im} \lambda$. Hence, any such B is determined by the parameter λ and $Lie(H^0)$ is spanned by

$$\begin{aligned} B_1(e^{-2}) &= -2e^{-2}, \quad B_1(e^{-1(10)}) = -e^{-1(10)}, \quad B_1(e^{0(10)}) = 0 \\ B_2(e^{-2}) &= 0, \quad B_2(e^{-1(10)}) = -ie^{-1(10)}, \quad B_2(e^{0(10)}) = -2ie^{0(10)}. \end{aligned}$$

By Table 1, $B_1 = \operatorname{ad}(E_1^0)|_{\mathfrak{m}}$, $B_2 = \operatorname{ad}(E_2^0)|_{\mathfrak{m}}$ and the first claim follows. The isomorphism between H^0 and a subgroup of $H \subset \operatorname{SO}_{3,2}^0$ follows by similar computations that determine explicitly the elements of $\operatorname{GL}_0^{\operatorname{gr}}(\mathfrak{m}, J)$ satisfying (5.3). \square

5.3. The flag of distributions of P^0 .

On P^0 , there is a natural flag of distributions $TP^0 = \mathcal{D}_{-2}^0 \supset \mathcal{D}_{-1}^0 \supset \mathcal{D}_{(0|-1)}^0 \supset \mathcal{D}_{(0|0)}^0 \supset \{0\}$, defined by

$$\mathcal{D}_{-1}^0 = (\pi_*^0)^{-1}(\mathcal{D}), \quad \mathcal{D}_{(0|-1)}^0 = (\pi_*^0)^{-1}(\mathcal{E}), \quad \mathcal{D}_{(0|0)}^0 = (\pi_*^0)^{-1}(\{0\}) = T^{\operatorname{Vert}}P^0. \quad (5.6)$$

These distributions determine the following filtrations

$$\mathcal{F} : \begin{matrix} T_y P^0 \\ = V_{-2} \end{matrix} \supset \begin{matrix} \mathcal{D}_{-1}^0|_y \\ = V_{-1} \end{matrix} \supset \begin{matrix} \mathcal{D}_{(0|-1)}^0|_y \\ = V_0 \end{matrix} \supset \{0\}, \quad (5.7)$$

$$\mathcal{F}_* : \begin{matrix} T_y P^0 \\ = V_{-2} \end{matrix} \supset \begin{matrix} \mathcal{D}_{-1}^0|_y \\ = V_{-1} \end{matrix} \supset \begin{matrix} \mathcal{D}_{(0|-1)}^0|_y \\ = V_0 \end{matrix} \supset \begin{matrix} \mathcal{D}_{(0|0)}^0|_y \\ = V_{(0|0)} \end{matrix} \supset \{0\}. \quad (5.8)$$

so that $\operatorname{gr}(T_y P^0, \mathcal{F}_*)$ is isomorphic to

$$\mathfrak{m} + \mathfrak{h}^0 = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0 + \mathfrak{h}^0 \subset \mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R}).$$

Any endomorphism $J_y : \mathcal{D}_{-1}^0|_y \rightarrow \mathcal{D}_{-1}^0|_y$, which projects onto the complex structure $J_x : \mathcal{D}|_x \rightarrow \mathcal{D}|_x$, $x = \pi^0(y)$, is a partial complex structure of $T_y P^0$ and two such partial complex structures are equivalent. Due to this, we may arbitrarily fix one such map J_y at any y and consider the linear frames

$$u_{\sharp} : \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + \mathfrak{m}^0 + \mathfrak{h}^0 \rightarrow V = T_y P^0 \quad (5.9)$$

that are adapted to the filtration and partial complex structures of $T_y P^0$. This property does not depend on the choice of J_y .

Recall that a frame $u_{\sharp} : \mathfrak{m} \rightarrow T_y P^0$ is completely determined by the corresponding basis of $T_y P^0$

$$f^{-2} = u_{\sharp}(e^{-2}), \quad f_j^{-1} = u_{\sharp}(e_j^{-1}), \quad f_j^0 = u_{\sharp}(e_j^0), \quad F_j^0 = u_{\sharp}(E_j^0).$$

Its dual coframe is indicated by $(f^{-2*}, \dots, F_2^{0*})$.

6. THE FIRST PROLONGATION OF THE TANAKA STRUCTURE P^0

6.1. Adapted frames of P^0 .

In the next definition, we denote by \widehat{E}_i^0 , $i = 1, 2$, the fundamental vector fields determined by the right actions of $E_i^0 \in \mathfrak{h}^0$ on P^0 .

Definition 6.1. Let $y \in P^0$ be a point over $x = \pi^0(y) \in M$. A linear frame $u_{\sharp} = (f^{-2}, f_i^{-1}, f_j^0, F_k^0)$, adapted to the filtration and partial complex structure of $T_y P^0$, is called *adapted frame of P^0* if

- i) $F_1^0 = \widehat{E}_1^0|_y$ and $F_2^0 = \widehat{E}_2^0|_y$;
- ii) the projected frame $\underline{u}_{\sharp} = (\underline{f}^{-2}, \underline{f}_i^{-1}, \underline{f}_j^0) = (\pi_*^0(f^{-2}), \pi_*^0(f_i^{-1}), \pi_*^0(f_j^0))$ of $T_x M$ belongs to the equivalence class $y = [\underline{u}_{\sharp}]$;

The collection $\mathcal{F}r_{1*}(P^0)$ of such frames is called *bundle of adapted frames of P^0* and we denote by $\pi_{1*} : \mathcal{F}r_{1*}(P^0) \rightarrow P^0$ the natural projection.

For any adapted frame $u_{\sharp} : \mathfrak{m} + \mathfrak{h}^0 \rightarrow T_y P^0$, let $u : \mathfrak{m} + \mathfrak{h}^0 \rightarrow \mathfrak{gr}(T_y P^0, \mathcal{F})$ be the corresponding isomorphism of graded vector spaces. By Remark 5.3, all frames $u_{\sharp} \in \mathcal{F}r_{1*}(P^0)|_y$ have the same associated isomorphism u , so that, by Lemma 4.3 and the remarks in §4.3, we have:

Lemma 6.2. *The triple $(\mathcal{F}r_{1*}(P^0), P^0, \pi_{1*})$ is a principal bundle of frames over P^0 , with structure group*

$$G_{\sharp}^{1*} = GL_{1*}(\mathfrak{m} + \mathfrak{h}^0, J) .$$

For any $y \in P^0$, the fiber $\mathcal{F}r_{1*}(P^0)|_y$ is in natural one-to-one correspondence with the collection of all s.h.s. $(H^{-2}, H^{-1}, H^{(0|-1)})$ of $V = T_y P^0$.

Later on, we will constantly identify an *adapted frame* at $y \in P^0$ with the corresponding *adapted s.h.s.* $H = (H^{-2}, H^{-1}, H^{(0|-1)})$ of $V = T_y P^0$. Moreover, given an adapted s.h.s. $H \subset T_y P^0$, associated with the linear frame $\widehat{H} = u_{\sharp}$, and an element $X \in \mathfrak{m} + \mathfrak{h}^0$, we denote

$$X_H = \widehat{H}(X) \in T_y P^0 .$$

Moreover, if $\widehat{H}' = \widehat{H} \cdot A$ for some $A \in GL_{1*}(\mathfrak{m} + \mathfrak{h}^0, J)$, we write

$$X_{H'} = A(X_H) , \quad \text{where we denote by } A(X_H) = \widehat{H} \circ A \circ \widehat{H}^{-1}(X_H) . \quad (6.1)$$

6.2. α -torsion, β -torsion and c -torsion.

Consider a smooth field of adapted s.h.s.

$$\mathcal{H} : \mathcal{U} \subset P^0 \rightarrow \{ \text{adapted s.h.s. in } T_{y'} P^0 , y' \in \mathcal{U} \} \simeq \mathcal{F}r_{1*}(P^0)|_{\mathcal{U}} ,$$

on a neighbourhood \mathcal{U} of $y \in P^0$. We call *torsion of \mathcal{H} at y* the bilinear map

$$\tau_{\mathcal{H}, y} \in \text{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{h}^0) , \quad \tau_{\mathcal{H}, y}(X, Y) = \widehat{\mathcal{H}}_y^{-1} \left([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y \right) . \quad (6.2)$$

Notice that the space $\text{Tor}(\mathfrak{m}) = \text{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{h})$, into which any torsion $\tau_{\mathcal{H},y}$ takes values, is naturally graded, with homogeneous subspaces

$$\text{Tor}^k(\mathfrak{m}) = \{\tau \in \text{Tor}(\mathfrak{m}) : \tau(\mathfrak{m}^i, \mathfrak{m}^j) \subset \mathfrak{g}^{i+j+k}, \text{ for all } i, j = -1, -2\}, \quad (6.3)$$

where we set $\mathfrak{g}^{-2} = \mathfrak{m}^{-2}$, $\mathfrak{g}^{-1} = \mathfrak{m}^{-1}$, $\mathfrak{g}^0 = \mathfrak{m}^0 + \mathfrak{h}^0$ and $\mathfrak{g}^i = \{0\}$ for any $i < -2$ or $i > 0$. Let us write $\tau_{\mathcal{H},y}$ as a sum of homogeneous components

$$\tau_{\mathcal{H},y} = \sum_k \tau_{\mathcal{H},y}^k \quad \text{with} \quad \tau_{\mathcal{H},y}^k \in \text{Tor}^k(\mathfrak{m}).$$

A priori, the torsion $\tau_{\mathcal{H},y}$ depends non only on $H = \mathcal{H}|_y$, but also on other parts of the first order jet of \mathcal{H} at y . Nonetheless, there are components of $\tau_{\mathcal{H},y}$ that depend only $H = \mathcal{H}|_y$. Such components are very important, because they allow to impose conditions, which are preserved by CR diffeomorphisms and determine canonical reductions of $\mathcal{F}r_{1*}(P^0)$. In the next definition, we name a few components that we later show to enjoy such crucial property.

Definition 6.3. Let \mathcal{H} be a local field of adapted s.h.s. on a neighbourhood $\mathcal{U} \subset P^0$ of y . We call

– α -torsion at y the map in $\text{Hom}(\mathfrak{m}^{-1} \times \mathfrak{m}^0, \mathfrak{m}^{-1})$

$$\alpha_{\mathcal{H},y} = \tau_{\mathcal{H},y}^0|_{\mathfrak{m}^{-1} \times \mathfrak{m}^0}; \quad (6.4)$$

– β -torsion at y the map in $\text{Hom}(\mathfrak{m}^{-1(10)} \times \mathfrak{m}^{0(01)}, \mathfrak{m}^{0(01)})$

$$\beta_{\mathcal{H},y} \stackrel{\text{def}}{=} (\tau_{\mathcal{H},y}^1|_{\mathfrak{m}^{-1(10)} \times \mathfrak{m}^{0(01)}})_{\mathfrak{m}^{0(01)}}, \quad (6.5)$$

where $(\cdot)_{\mathfrak{m}^{0(01)}}$ denotes the component in the antiholomorphic subspace of $\mathfrak{m}^{0\mathbb{C}} = \mathfrak{m}^{0(10)} + \mathfrak{m}^{0(01)}$;

– c -torsion at y the map in $\text{Hom}(\Lambda^2(\mathfrak{m}^{-2} + \mathfrak{m}^{-1}), \mathfrak{m} + \mathfrak{h}^0)$

$$c_{\mathcal{H},y} = \tau_{\mathcal{H},y}|_{\Lambda^2(\mathfrak{m}^{-2} + \mathfrak{m}^{-1})}. \quad (6.6)$$

We denote by $c_{\mathcal{H},y}^k$ the homogeneous components $c_{\mathcal{H},y}^k = \tau_{\mathcal{H},y}^k|_{\Lambda^2(\mathfrak{m}^{-2} + \mathfrak{m}^{-1})}$.

6.3. Strongly adapted frames of P^0 and the first prolongation P^1 .

6.3.1. *A preliminary step: the reduction $\widetilde{\mathcal{F}r}_1(P^0)$ of $\mathcal{F}r_{1*}(P^0)$.*

As before, \mathcal{H} is a field of adapted s.h.s. on a neighbourhood of $y \in P^0$.

Lemma 6.4. *The α -torsion $\alpha_{\mathcal{H},y}$ depends only on $H = \mathcal{H}|_y$ in $T_y P^0$ and can be considered as a tensor α_H , naturally associated with $\widehat{H} \in \mathcal{F}r_{1*}(P^0)$.*

The collection $\mathcal{F}r_1(P^0) \subset \mathcal{F}r_{1}(P^0)$ of adapted frames \widehat{H} such that*

$$\left(\alpha_H(e^{-1(10)}, e^{0(10)}) \right)_{\mathfrak{m}^{-1(10)}} = 0 = \left(\alpha_H(e^{-1(01)}, e^{0(10)}) \right)_{\mathfrak{m}^{-1(01)}}, \quad (6.7)$$

is a principal subbundle with structure group $\text{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$.

Proof. Let $\mathcal{H}, \mathcal{H}'$ be two fields of adapted s.h.s. with $\mathcal{H}|_y = \mathcal{H}'|_y = H$. Then there exists a map

$$A = I + B : \mathcal{U} \longrightarrow \mathrm{GL}_{1*}(\mathfrak{m} + \mathfrak{h}^0, J), \quad B_{y'} \in \mathfrak{gl}_{1*}(\mathfrak{m} + \mathfrak{h}^0, J), \quad y' \in \mathcal{U},$$

with $B_y = 0$ and such that $\hat{\mathcal{H}}'_{y'} = \hat{\mathcal{H}}_{y'} \circ A_{y'}$ for any y' of a neighbourhood of y . It follows that, for any $X, Y \in \mathfrak{m} + \mathfrak{h}^0$,

$$[X_{\mathcal{H}'}, Y_{\mathcal{H}'}]|_y = [X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y + (X_{\mathcal{H}} \cdot B|_y)(Y)_{\mathcal{H}} - (Y_{\mathcal{H}} \cdot B|_y)(X)_{\mathcal{H}}, \quad (6.8)$$

where we denoted by $X_{\mathcal{H}} \cdot B|_y, Y_{\mathcal{H}} \cdot B|_y$ the linear maps

$$X_{\mathcal{H}} \cdot B|_y = \left. \frac{dB(\gamma_s)}{ds} \right|_{s=0}, \quad Y_{\mathcal{H}} \cdot B|_y = \left. \frac{dB(\eta_s)}{ds} \right|_{s=0},$$

for some curves γ_s, η_s with $\gamma_0 = \eta_0 = y$ and $\dot{\gamma}_0 = X_{\mathcal{H}}|_y, \dot{\eta}_0 = Y_{\mathcal{H}}|_y$. Notice that $X_{\mathcal{H}} \cdot B|_y, Y_{\mathcal{H}} \cdot B|_y \in \mathfrak{gl}_{1*}(\mathfrak{m} + \mathfrak{h}^0, J)$. So, if $X \in \mathfrak{m}^{-1}, Y \in \mathfrak{m}^0$,

$$\alpha_{\mathcal{H}', y}(X, Y) = \left(\hat{\mathcal{H}}'^{-1}([X_{\mathcal{H}'}, Y_{\mathcal{H}'}]|_y) \right)_{\mathfrak{m}^{-1}} = \left(\hat{\mathcal{H}}_y^{-1}([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y) \right)_{\mathfrak{m}^{-1}} = \alpha_{\mathcal{H}, y}(X, Y)$$

proving the first claim. For the second claim, we need to show that:

- a) for any $y \in P^0$, there exists an adapted s.h.s. H in $T_y P^0$ for which (6.7) holds;
- b) two such adapted s.h.s. are related one to the other by an element in $\mathrm{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$.

Consider a fixed s.h.s. H_o in $T_y P^0$ and a field of s.h.s. \mathcal{H}_o on a neighbourhood of y with $\mathcal{H}_o|_y = H_o$. Let also H a second s.h.s. in $T_y P^0$ and a field of s.h.s. \mathcal{H} with $\mathcal{H}|_y = H$. As before, we have that $\hat{\mathcal{H}}|_{y'} = \hat{\mathcal{H}}_o|_{y'} \circ A_{y'}$, $y' \in \mathcal{U}$, for some map $A = I + B$ with values in $\mathrm{GL}_{1*}(\mathfrak{m} + \mathfrak{h}^0, J)$. By definitions of the distributions $\mathcal{D}_{(0|-1)}^0 \subset \mathcal{D}_{(-1)}^0$ and of the adapted frames, for any $X \in \mathfrak{m}^{-1}, Y \in \mathfrak{m}^0$, the vector $\alpha_H(X, Y)$ is equal to

$$\begin{aligned} & \left(\hat{H}^{-1}([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y) \right)_{\mathfrak{m}^{-1}} = \left(A^{-1} \circ \hat{H}_o^{-1}([X_{\mathcal{H}_o} + B(X_{\mathcal{H}_o}), Y_{\mathcal{H}_o} + B(Y_{\mathcal{H}_o})]|_y) \right)_{\mathfrak{m}^{-1}} \\ & = \alpha_{H_o}(X, Y) + \left((I + B)^{-1} \left(\hat{H}_o^{-1}([X_{\mathcal{H}_o}, B(Y_{\mathcal{H}_o})] \bmod \mathcal{D}_{(0|-1)}^0) \right) \right)_{\mathfrak{m}^{-1}} = \\ & = \alpha_{H_o}(X, Y) + \left(\hat{H}_o^{-1}([X_{\mathcal{H}_o}, B(Y_{\mathcal{H}_o})]|_y) \right)_{\mathfrak{m}^{-1}}. \end{aligned} \quad (6.9)$$

Since B takes values in $\mathfrak{gl}_{1*}(\mathfrak{m} + \mathfrak{h}^0, J)$, one has that, modulo terms of higher grade, B_y is of the form

$$\begin{aligned} B_y(e^{-2}) &= \lambda e^{-1(10)} + \bar{\lambda} e^{-1(01)}, \quad B_y(e^{-1(10)}) = \mu e^{0(10)}, \\ B_y(e^{0(10)}) &= \nu E^{0(10)} + \nu' E^{0(01)}, \quad \text{for some } \lambda, \mu, \nu, \nu' \in \mathbb{C}. \end{aligned} \quad (6.10)$$

Therefore, by (6.3.1) we have that

$$\begin{aligned} \alpha_H(e^{-1(10)}, e^{0(10)}) &= \alpha_{H_o}(e^{-1(10)}, e^{0(10)}) + \\ &+ \left(\hat{\mathcal{H}}_o^{-1} \left(\nu [e_{\mathcal{H}_o}^{-1(10)}, E_{\mathcal{H}_o}^{0(10)}]|_y + \nu' [e_{\mathcal{H}_o}^{-1(10)}, E_{\mathcal{H}_o}^{0(01)}]|_y \right) \right)_{\mathfrak{m}^{-1}} = \\ &= \alpha_{H_o}(e^{-1(10)}, e^{0(10)}) + \nu e^{-1(10)} \end{aligned} \quad (6.11)$$

$$\begin{aligned}
\alpha_H(e^{-1(01)}, e^{0(10)}) &= \alpha_{H_o}(e^{-1(01)}, e^{0(10)}) + \\
&+ \left(\widehat{\mathcal{H}}_o^{-1} \left(\nu \left[e_{\mathcal{H}_o}^{-1(01)}, E_{\mathcal{H}_o}^{0(10)} \right] \Big|_y + \nu' \left[e_{\mathcal{H}_o}^{-1(01)}, E_{\mathcal{H}_o}^{0(01)} \right] \Big|_y \right) \Big|_{\mathfrak{m}^{-1}} \right) = \\
&= \alpha_{H_o}(e^{-1(01)}, e^{0(10)}) + \nu' e^{-1(01)} , \tag{6.12}
\end{aligned}$$

where we used the fact that $E_{\mathcal{H}_o}^{0(10)}$ and $E_{\mathcal{H}_o}^{0(01)}$ are fundamental vector fields. From (6.3.1) and (6.3.1), one can directly see that there always exist ν, ν' such that H satisfies (6.7) and that two given s.h.s. H, H' satisfy (6.7) if and only if their corresponding adapted frames are related by a transformation $A = I + B$, in which B is of the form (6.3.1) with $\nu, \nu' = 0$. Since this is equivalent to say that $B \in \mathfrak{gl}_1(\mathfrak{m} + \mathfrak{h}^0, J)$, we get that $\mathcal{F}r_1(P^0)$ is a $\mathrm{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$ -reduction. \square

From now on, we consider only adapted s.h.s. and fields of adapted s.h.s., whose corresponding adapted frames are in the reduction $\mathcal{F}r_1(P^0)$.

Lemma 6.5. *If we restrict to $\mathcal{F}r_1(P^0)$, for any field \mathcal{H} of adapted s.h.s. around $y \in P^0$, the β -torsion $\beta_{\mathcal{H}, y}$ depends only on $H = \mathcal{H}|_y$ and can be considered as a tensor β_H , naturally associated with $\widehat{H} \in \mathcal{F}r_1(P^0)|_y$.*

The collection $\widetilde{\mathcal{F}r}_1(P^0) \subset \mathcal{F}r_1(P^0)$ of frames with $\beta_H = 0$ is a subbundle with structure group

$$G_{\sharp}^1 = L^1 \ltimes \mathrm{GL}_2(\mathfrak{m} + \mathfrak{h}^0, J) \subset \mathrm{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$$

where L^1 denotes the group of linear isomorphisms

$$\begin{aligned}
L^1 = \{ I + B \quad : \quad B \in \mathfrak{gl}_1^{\mathrm{gr}}(\mathfrak{m} + \mathfrak{h}^0, J) \text{ and } (B(e^{-1(10)}))_{\mathfrak{m}^{0(10)}} = \mu e^{0(10)} , \\
(B(e^{-1(10)}))_{\mathfrak{h}^0 \mathbb{C}} = \nu E^{0(10)} + (\nu - \overline{\mu}) E^{0(01)} \text{ for some } \nu, \mu \in \mathbb{C} \} . \tag{6.13}
\end{aligned}$$

Proof. The first claim is proved as in Lemma 6.4. In fact, let $\mathcal{H}, \mathcal{H}'$ be two fields of s.h.s. in $\mathcal{F}r_1(P^0)$, with $\mathcal{H}|_y = \mathcal{H}'|_y = H$, and denote by $A = I + B$ a $\mathrm{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$ -valued map such that $\widehat{\mathcal{H}}' = \widehat{\mathcal{H}} \circ A$. Using the same notation of (6.8), for any $X \in \mathfrak{m}^{-1(10)}, Y \in \mathfrak{m}^{0(01)}$, we have that $X_{\mathcal{H}} \cdot B|_y, Y_{\mathcal{H}} \cdot B|_y$ are in $\mathfrak{gl}_1(\mathfrak{m} + \mathfrak{h}^0, J)$ and

$$(Y_{\widehat{\mathcal{H}}} \cdot B|_y(X))_{\mathcal{H}} \in \widehat{H}(\mathfrak{m}^{0(10)} + (\mathfrak{h}^0)^{\mathbb{C}}) \quad , \quad (X_{\widehat{\mathcal{H}}} \cdot B|_y(Y))_{\mathcal{H}} = 0 .$$

From this and (6.8), it follows that $(\tau_{\mathcal{H}', y}^1(X, Y))_{\mathfrak{m}^{0(01)}} = (\tau_{\mathcal{H}, y}^1(X, Y))_{\mathfrak{m}^{0(01)}}$, so that $\beta_{\mathcal{H}', y} = \beta_{\mathcal{H}, y}$.

Also the second claim is proved as in Lemma 6.4. Consider a fixed s.h.s. H_o with $\widehat{H}_o \in \mathcal{F}r_1(P^0)|_y$ and let \mathcal{H}_o be a field of s.h.s., with associated frames in $\mathcal{F}r_1(P^0)$, such that $\mathcal{H}_o|_y = H_o$. Take also a second s.h.s. H in $\mathcal{F}r_1(P^0)|_y$ and a field of s.h.s. \mathcal{H} with $\mathcal{H}|_y = H$, with corresponding frames in $\mathcal{F}r_1(P^0)$. Finally, let $A = I + B$ be a $\mathrm{GL}_1(\mathfrak{m} + \mathfrak{h}^0, J)$ -valued map such that $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_o \circ A$.

Since $\pi_*^0(\mathcal{D}_{(0|-1)}^0) = \mathcal{E}$ is an integrable complex distribution of complex dimension one, there is no loss of generality if we assume that the (locally defined) fields of s.h.s. \mathcal{H} , \mathcal{H}' are such that

$$\pi_*^0([e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{0(01)}]) = 0 = \pi_*^0([e_{\mathcal{H}}^{0(10)}, e_{\mathcal{H}}^{0(01)}]) \quad (6.14)$$

or, equivalently, that the vector fields $[e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{0(01)}]$, $[e_{\mathcal{H}}^{0(10)}, e_{\mathcal{H}}^{0(01)}]$ take values in $\mathcal{D}_{(0|0)}^0 = T^{\text{Vert}}P^0$. Since these frames satisfy (6.7), we also have that

$$[e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{-1(01)}] = \rho e_{\mathcal{H}_o}^{-1(10)} \mod \mathcal{D}_{(0|-1)}^0 \quad (6.15)$$

for some suitable complex function ρ . Moreover, by (5.2),

$$[e_{\mathcal{H}_o}^{-1(10)}, e_{\mathcal{H}_o}^{-1(01)}] = \frac{i}{2} e_{\mathcal{H}_o}^{-2} \mod \mathcal{D}_{-1}^0 .$$

$$[[e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{-1(01)}], e_{\mathcal{H}_o}^{-1(01)}] = -\frac{i}{2} e_{\mathcal{H}_o}^{-2} \mod \mathcal{D}_{-1}^0 .$$

This and (6.15) imply that $\rho \equiv -1$ and that

$$[e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{-1(01)}] = -e_{\mathcal{H}_o}^{-1(10)} = [e^{0(10)}, e^{-1(01)}]_{\mathcal{H}_o} \mod \mathcal{D}_{(0|-1)}^0 . \quad (6.16)$$

These arguments hold for any field of s.h.s. in $\mathcal{F}r_1(P^0)$ and imply that (6.16) holds for \mathcal{H} as well. From (6.8), (6.14) and (6.16) one gets

$$\begin{aligned} \beta_H(e^{-1(10)}, e^{0(10)}) &= \left(\widehat{\mathcal{H}}^{-1} \left([e_{\mathcal{H}}^{-1(10)}, e_{\mathcal{H}}^{0(10)}] \Big|_y \right) \right)_{\mathfrak{m}^{0(01)}} = \\ &= \left((I + B_y)^{-1} \circ \widehat{\mathcal{H}}_o^{-1} \left([e_{\mathcal{H}}^{-1(10)}, e_{\mathcal{H}}^{0(10)}] \Big|_y \right) \right)_{\mathfrak{m}^{0(01)}} = \\ &= \beta_{H_o}(e^{-1(10)}, e^{0(10)}) - \left(B_y([e^{-1(10)}, e^{0(01)}]) \right)_{\mathfrak{m}^{0(01)}} + \left([B_y(e^{-1(10)}), e^{0(01)}] \right)_{\mathfrak{m}^{0(01)}} \end{aligned} \quad (6.17)$$

Since B takes values in $\mathfrak{gl}_1(\mathfrak{m} + \mathfrak{h}^0, J)$, the images under the linear map B_y are vectors of the form (modulo terms of higher grade)

$$\begin{aligned} B_y(e^{-2}) &= \lambda e^{-1(10)} + \bar{\lambda} e^{-1(01)} , \quad B_y(e^{-1(10)}) = \mu e^{0(10)} + \nu E^{0(10)} + \nu' E^{0(01)} , \\ B_y(e^{-1(01)}) &= \bar{\mu} e^{0(01)} + \bar{\nu} E^{0(01)} + \bar{\nu}' E^{0(10)} , \quad B_y(e^{0(10)}) = B(e^{0(01)}) = 0 \end{aligned} \quad (6.18)$$

for some $\lambda, \mu, \nu, \nu' \in \mathbb{C}$. So, by Table 1, (6.3.1) is equivalent to

$$\beta_H(e^{-1(10)}, e^{0(10)}) = \beta_{H_o}(e^{-1(10)}, e^{0(10)}) + (-\bar{\mu} + \nu - \nu') e^{0(01)} . \quad (6.19)$$

This shows that if $\nu' = \nu - \bar{\mu} - \beta_{H_o}(e^{-1(10)}, e^{0(10)})$, then $\beta_{\mathcal{H}} = 0$, so that $\widehat{\mathcal{F}}r_1(P^0)|_y \neq \emptyset$. From (6.19), it follows also that $\widehat{\mathcal{F}}r_1(P^0)$ is a reduction of $\mathcal{F}r_1(P^0)$ with structure group G_{\sharp}^1 . \square

6.3.2. The strongly adapted frames of P^0 .

From now on, we limit ourselves to the bundle $\pi_1 : \widetilde{\mathcal{F}r}_1(P^0) \rightarrow P^0$ and the s.h.s. H 's or fields of adapted s.h.s. \mathcal{H} are assumed to correspond to frames in $\widetilde{\mathcal{F}r}_1(P^0)$. Such frames are shortly call *nicely adapted*.

Lemma 6.6. *For any field \mathcal{H} of nicely adapted s.h.s. around $y \in P^0$, the k -th components $\tau_{\mathcal{H},y}^k$ and $c_{\mathcal{H},y}^k$ of torsion and c -torsion are such that:*

- (i) $\tau_{\mathcal{H},y}^k = 0$ for any $k < 0$;
- (ii) $\tau_{\mathcal{H},y}^0(X, Y) = [X, Y]$ for any $X \in \mathfrak{m}^{-2} + \mathfrak{m}^{-1}$ and $Y \in \mathfrak{m}$;
- (iii) $c_{\mathcal{H},y}^1$ depends only on the s.h.s. $H = \mathcal{H}|_y$ and it can be considered as a tensor c_H^1 , associated with $\widehat{H} \in \widetilde{\mathcal{F}r}_1(P^0)|_y$.

Proof. For what concerns (i) and (ii), the only statement that does not follows directly from (5.2) and definitions is claim (ii) for the case $X \in \mathfrak{m}^{-2} + \mathfrak{m}^{-1}$ and $Y \in \mathfrak{m}^0$. Assume that $X \in \mathfrak{m}^{-1}$, $Y \in \mathfrak{m}^0$. Then (6.7) and (6.16) imply that

$$[X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y = ([X, Y]_{\mathcal{H}})|_y \mod \mathcal{D}_{(0|-1)}^0|_y ,$$

from which it follows $\tau_{\mathcal{H},y}^0(X, Y) = [X, Y]$. On the other hand, when $X = e^{-2}$ and $Y = e^{0(10)}$, we have that $e_{\mathcal{H}}^{-2} = -2i[e_{\mathcal{H}}^{-1(10)}, e_{\mathcal{H}}^{-1(01)}] \mod \mathcal{D}_{(0|-1)}^0$, so that, by (6.7) and (6.16),

$$[e_{\mathcal{H}}^{-2}, e_{\mathcal{H}}^{0(10)}] = -2i[[e_{\mathcal{H}}^{-1(10)}, e_{\mathcal{H}}^{-1(01)}], e_{\mathcal{H}}^{0(10)}] \mod \mathcal{D}_{-1}^0 = 0 \mod \mathcal{D}_{-1}^0 .$$

This shows that $\tau_{\mathcal{H},y}^0(e^{-2}, e^{0(10)}) = 0 = [e^{-2}, e^{0(10)}]$ and (ii) follows.

Now, as in the proof of Lemmas 6.4 and 6.5, consider a fixed nicely adapted s.h.s. H_o and a field \mathcal{H}_o of nicely adapted s.h.s. around y with $\mathcal{H}_o|_y = H_o$. Take also a second nicely adapted s.h.s. H and a field of nicely adapted s.h.s. \mathcal{H} with $\mathcal{H}|_y = H$ and denote by $A = I + B$ the G_{\sharp}^1 -valued map such that $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_o \circ A$. Consider the expression (6.8) for the Lie bracket $[X_{\mathcal{H}'}, Y_{\mathcal{H}'}]|_y$ in the case $X \in \mathfrak{m}^i$, $Y \in \mathfrak{m}^j$, $i, j \in \{-1, -2\}$. Since B takes values in $\mathfrak{l}^1 + \mathfrak{gl}_2(\mathfrak{m} + \mathfrak{h}^0, J)$ with $\mathfrak{l}^1 = Lie(L^1)$, it follows that $X_{\mathcal{H}} \cdot B|_y$ and $Y_{\mathcal{H}} \cdot B|_y$ are in $\mathfrak{l}^1 + \mathfrak{gl}_2(\mathfrak{m} + \mathfrak{h}^0, J)$ and the last two terms of (6.8) take value in $\mathcal{D}_{j+1}^0|_y$ and $\mathcal{D}_{i+1}^0|_y$, respectively. Since $i+1, j+1 > i+j+1$, it follows that $c_{\mathcal{H}',y}^1(X, Y) = \left(\widehat{H}^{-1} \left([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y \right) \right)_{\mathfrak{m}^{i+j+1}} = c_{\mathcal{H},y}^1(X, Y)$. \square

Recall now that, for a graded Lie algebra \mathfrak{n} and a graded \mathfrak{n} -module W , the space of skew-symmetric multi-linear maps

$$\mathcal{C}_k^{\ell}(\mathfrak{n}, W) = \{ c \in \text{Hom}(\Lambda^{\ell} \mathfrak{n}, W) : c(\mathfrak{n}^{i_1} \wedge \dots \wedge \mathfrak{n}^{i_{\ell}}) \subset W^{i_1 + \dots + i_{\ell} + k} \} .$$

is called *homogeneous space of ℓ -cochains of degree k* . Its *differential* is the coboundary operator $\partial : C_k^\ell(\mathfrak{n}, W) \longrightarrow C_k^{\ell+1}(\mathfrak{n}, W)$

$$\begin{aligned} \partial c(X_1 \wedge \cdots \wedge X_{\ell+1}) &= \sum_s (-1)^{s+1} X_s \cdot c(X_1 \wedge \cdots \hat{}_s \cdots \wedge X_{\ell+1}) + \\ &\quad + \sum_{s < t} (-1)^{s+t} c([X_s, X_t] \wedge X_1 \wedge \cdots \hat{}_s \cdots \hat{}_t \cdots \wedge X_{\ell+1}) . \end{aligned}$$

and the corresponding cohomology spaces are denoted by

$$H_k^\ell(\mathfrak{n}, W) = \frac{\text{Ker } \partial|_{C_k^\ell(\mathfrak{n}, W)}}{\partial(C_k^{\ell-1}(\mathfrak{n}, W))} .$$

Let $\mathfrak{m}_- = \mathfrak{m}^{-2} + \mathfrak{m}^{-1}$ and $\mathfrak{h}_+ = \sum_{i \geq 0} \mathfrak{h}^i$. Consider $\mathfrak{g} = \mathfrak{so}_{3,2}(\mathbb{R}) = \mathfrak{m}^{-2} + \mathfrak{m}^{-1} + (\mathfrak{m}^0 + \mathfrak{h}^0) + \mathfrak{h}^1 + \mathfrak{h}^2$ as a graded \mathfrak{m}_- -module and notice that

$$\mathfrak{gl}_k(\mathfrak{m}_- + \mathfrak{m}^0 + \mathfrak{h}^0 + \sum_{i=1}^{k-1} \mathfrak{h}^i) \simeq C_k^1(\mathfrak{m}_-, \mathfrak{g}) , \quad \text{Tor}^k(\mathfrak{m}) \simeq C_k^2(\mathfrak{m}_-, \mathfrak{g}) .$$

Hence, the differential ∂ gives a map from $\mathfrak{gl}_k(\mathfrak{m}_- + \sum_{i=0}^{k-1} \mathfrak{g}^i)$ into $\text{Tor}^k(\mathfrak{m})$. Recall also that, via the Killing form of \mathfrak{g} , each space $C_k^\ell(\mathfrak{m}_-, \mathfrak{g})$ can be identified with $C_k^\ell(\mathfrak{h}_+, \mathfrak{g}^*)$ and the opposite of the map $\partial : C_k^\ell(\mathfrak{h}_+, \mathfrak{g}^*) \longrightarrow C_k^{\ell+1}(\mathfrak{h}_+, \mathfrak{g}^*)$ can be identified with a linear map

$$\partial^* : C_k^\ell(\mathfrak{m}_-, \mathfrak{g}) \longrightarrow C_k^{\ell-1}(\mathfrak{m}_-, \mathfrak{g}) ,$$

called *codifferential*. By Kostant's theory ([Ko]), for any $\ell, k \geq 0$, we have the $\text{ad}(\mathfrak{g}^0)$ -invariant direct sum decomposition

$$C_k^\ell(\mathfrak{m}_-, \mathfrak{g}) = \partial C_k^{\ell-1}(\mathfrak{m}_-, \mathfrak{g}) \oplus H_k^\ell(\mathfrak{m}_-, \mathfrak{g}) \oplus \partial^* C_k^{\ell+1}(\mathfrak{m}_-, \mathfrak{g}) , \quad (6.20)$$

from which it follows the $(\mathfrak{h}_+ + \mathfrak{g}^0)$ -invariance of the spaces

$$\ker \partial^*|_{C_k^\ell(\mathfrak{m}_-, \mathfrak{g})} = \sum_k \left(H_k^\ell(\mathfrak{m}_-, \mathfrak{g}) + \partial^* C_k^{\ell+1}(\mathfrak{m}_-, \mathfrak{g}) \right)$$

(we consider $\mathfrak{h}_+ + \mathfrak{g}^0$ acting on $\mathfrak{m}_- \simeq \mathfrak{g}/(\mathfrak{h}_+ + \mathfrak{g}^0)$ with the adjoint action).

Consider now the Lie algebra $\mathfrak{l}^1 = \text{Lie}(L^1)$ and the complementary subspace of $\partial \mathfrak{l}^1$ in $\text{Tor}^1(\mathfrak{m})$, defined as follows. First of all, notice that the restriction $\text{ad}(E_2^0)|_{\mathfrak{g}^i}$ on each graded space $\mathfrak{g}^i = \mathfrak{m}^i$ or \mathfrak{h}^i is either trivial or equal to a multiple of $J|_{\mathfrak{g}^i}$. This means that $K = e^{\mathbb{R} \text{ad}(E_2^0)|_{\mathfrak{g}^i}}$ is isomorphic to S^1 and we may consider a K -invariant Euclidean inner product \langle, \rangle on $\partial C_1^1(\mathfrak{m}_-, \mathfrak{g})$. On the other hand, from definitions, one can directly check that \mathfrak{l}^1 is $\text{ad}(E_2^0)$ -invariant and, consequently, its orthogonal complement $(\partial \mathfrak{l}^1)^\perp$ in $\partial C_1^1(\mathfrak{m}_-, \mathfrak{g})$ is $\text{ad}(E_2^0)$ -invariant as well. The space $(\mathfrak{l}^1)^\perp$ is also $\text{ad}(E_1^0)$ -invariant, because the action of E_1^0 on $\partial C_1^1(\mathfrak{m}_-, \mathfrak{g})$ is equal to the identity: in fact, E_1^0 is a grading element and $\partial C_1^1(\mathfrak{m}_-, \mathfrak{g})$ is a homogeneous space of grade $+1$. So, $(\partial \mathfrak{l}^1)^\perp$ is \mathfrak{h}^0 -invariant and complementary to $\partial \mathfrak{l}^1$

in $\partial C_1^1(\mathfrak{m}_-, \mathfrak{g})$, while $(\partial \mathfrak{l}^1)^\perp + \ker \partial^*|_{C_1^2(\mathfrak{m}_-, \mathfrak{g})}$ is $\text{ad}(\mathfrak{h}^0)$ -invariant and complementary to $\partial \mathfrak{l}^1$ in $\text{Tor}^1(\mathfrak{m}) \simeq C_1^2(\mathfrak{m}_-, \mathfrak{g})$. This observation allows the following

Definition 6.7. We call *strongly adapted frame* of $T_y P^0$ a nicely adapted frame $\widehat{H} \in \widetilde{\mathcal{F}r}_1(P^0)|_y$, with c -torsion such that

$$c_H^1 \in (\partial \mathfrak{l}^1)^\perp + \ker \partial^*|_{C_1^2(\mathfrak{m}_-, \mathfrak{g})} . \quad (6.21)$$

Lemma 6.8. The collection $P_\#^1 \subset \widetilde{\mathcal{F}r}_1(P^0)$ of strongly adapted frames is a reduction with structure group $H_\#^1 = H^1 \ltimes \text{GL}_2(\mathfrak{m} + \mathfrak{h}^0)$, where

$$H^1 = \{ I + B \in G_\#^1 : \partial B = 0 \} . \quad (6.22)$$

Proof. Let H_o be a nicely adapted s.h.s. in $T_y P^0$ and H some other nicely adapted s.h.s. in the same tangent space. As usual, we consider two fields of nicely adapted s.h.s. $\mathcal{H}_o, \mathcal{H}$ with $\mathcal{H}_o|_y = H_o, \mathcal{H}|_y = H$ and a local map $A = I + B : \mathcal{U} \rightarrow G_\#^1$ such that $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_o \circ A$. Using definitions, Lemma 6.6 (i), (ii) and standard arguments, for $X \in \mathfrak{m}^i, Y \in \mathfrak{m}^j, i, j \in \{-2, -1\}$,

$$\begin{aligned} c_H^1(X, Y) &= \left((I + B)^{-1} \left(\widehat{H}_o^{-1}([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_y) \right) \right)_{\mathfrak{m}^{i+j+1}} = \\ &= c_{H_o}^1(X, Y) - (B(\tau_{\mathcal{H}_o, t}^0(X, Y)))_{\mathfrak{m}^{i+j+1}} + (\tau_{\mathcal{H}_o, y}^0(B(X), Y))_{\mathfrak{m}^{i+j+1}} + \\ &\quad + (\tau_{\mathcal{H}_o, y}^0(X, B(Y)))_{\mathfrak{m}^{i+j+1}} = c_{H_o}^1(X, Y) - \partial B(X, Y) . \end{aligned} \quad (6.23)$$

Hence, if we denote by $c_H^1 = (c_H^1)_{\partial \mathfrak{l}^1} + (c_H^1)_{(\partial \mathfrak{l}^1)^\perp + \ker \partial^*|_{C_1^2(\mathfrak{m}_-, \mathfrak{g})}}$ the decomposition of c_H^1 into a sum of elements in $\partial \mathfrak{l}^1$ and $(\partial \mathfrak{l}^1)^\perp + \ker \partial^*|_{C_1^2(\mathfrak{m}_-, \mathfrak{g})}$, we see that there always exists B such that $(c_H^1)_{\partial \mathfrak{l}^1} = 0$, proving that the fiber of $P_\#^1$ over y is not empty. The equality (6.3.2) shows also that nicely adapted frames $\widehat{H}, \widehat{H}'$ are both in $P_\#^1$ if and only if $\widehat{H}' = \widehat{H} \circ A$ for some $A = I + B$, with $\partial B = 0$. This implies that $P_\#^1$ has $H^1 \ltimes \text{GL}_2(\mathfrak{m} + \mathfrak{h}^0, J)$ as structure group. Note also that $\text{GL}_2(\mathfrak{m} + \mathfrak{h}^0, J) = \text{GL}_2(\mathfrak{m} + \mathfrak{h}^0)$. \square

Lemma 6.9. The Lie algebra $\text{Lie}(H^1)$ has dimension 2 and is equal to

$$\text{Lie}(H^1) = \text{ad}(\mathfrak{h}^1)|_{\mathfrak{m}} ,$$

where $\text{ad}(\mathfrak{h}^1)|_{\mathfrak{m}} \subset \mathfrak{gl}_1(\mathfrak{m}, J)$ denotes the space of the adjoint actions $\text{ad}(X)|_{\mathfrak{m}}$ with $X \in \mathfrak{h}^1 \subset \mathfrak{so}_{3,2}(\mathbb{R})$. In particular, $\text{Lie}(H^1) \simeq \mathfrak{h}^1$ and is naturally endowed with the complex structure $J|_{\mathfrak{h}^1}$ defined in (3.10).

Proof. Recall that $B \in \mathfrak{l}^1 = \text{Lie}(L^1)$ if and only if, modulo terms of higher grade, it is of the form (6.3.1) for some $\nu' = \nu - \bar{\mu}$. On the other hand, condition $\partial B = 0$ is equivalent to

$$\left(B([e^{-2}, e^{-1(10)}]) \right)_{\mathfrak{m}^{-2}} = \left([B(e^{-2}), e^{-1(10)}] + [e^{-2}, B(e^{-1(10)})] \right)_{\mathfrak{m}^{-2}} ,$$

$$\left(B([e^{-1(10)}, e^{-1(01)}]) \right)_{\mathfrak{m}^{-1}} = \left([B(e^{-1(10)}), e^{-1(01)}] + [e^{-1(10)}, B(e^{-1(01)})] \right)_{\mathfrak{m}^{-1}}.$$

Using Table 1, one can check that these conditions correspond to require

$$\nu = \frac{i}{2}\bar{\lambda}, \quad \mu = -\frac{i}{2}\lambda, \quad \nu' = 0,$$

from which it follows that $\text{Lie}(H^1)$ is generated by the maps B_1, B_2 corresponding to $\lambda = 1$ and $\lambda = i$, i.e.,

$$\begin{aligned} B_1(e^{-2}) &= e^{-1(10)} + e^{-1(01)} = \text{ad}(-E_2^1)(e_{-2}), \\ B_1(e^{-1(10)}) &= -\frac{i}{2}e^{0(10)} + \frac{i}{2}E^{0(10)} = \text{ad}(-E_2^1)(e^{-1(10)}), \\ B(e^{0(10)}) &= 0 = \text{ad}(-E_2^1)(e^{-1(10)}) \bmod \mathfrak{g}^1, \\ B_2(e^{-2}) &= i(e^{-1(10)} - e^{-1(01)}) = \text{ad}(E_1^1)(e_{-2}), \\ B(e^{-1(10)}) &= \frac{1}{2}e^{0(10)} + \frac{1}{2}E^{0(10)} = \text{ad}(E_1^1)(e^{-1(10)}), \\ B(e^{0(10)}) &= 0 = \text{ad}(E_1^1)(e^{-1(10)}) \bmod \mathfrak{g}^1, \end{aligned} \tag{6.24}$$

$$\tag{6.25}$$

and this concludes the proof. \square

The quotient bundle

$$\pi^1 : P^1 = P_{\#}^1 / GL_2(\mathfrak{m} + \mathfrak{h}^0) \longrightarrow P^0$$

with π^1 induced by the natural projection $\pi_{\#}^1 : P_{\#}^1 \longrightarrow P^0$, is called *first prolongation of the Tanaka structure* $\pi^0 : P^0 \longrightarrow M$. It is a principal bundle over P^0 , but it also a principal bundle over M . In fact,

Lemma 6.10. *There exists a natural right action of a semidirect product $H^0 \ltimes H^1$ on the bundle $\pi^0 \circ \pi^1 : P^1 \longrightarrow M$, which makes it a principal bundle over M , canonically associated with the girdled CR structure (\mathcal{D}, J) .*

The Lie group $H^0 \ltimes H^1$ is isomorphic to H/H^2 , where we denote by H^2 the connected subgroup of $H = \text{Aut}_{x_o}(M_o) \subset \text{SO}_{3,2}$ with subalgebra $\text{Lie}(H^2) = \mathfrak{h}^2$.

Proof. For any $h \in H^0$, we denote by $f_h : P^0 \longrightarrow P^0$ the diffeomorphism determined by the right action on P^0 , and by $\widehat{f}_h : \mathcal{F}r(P_0) \longrightarrow \mathcal{F}r(P_0)$ the associated diffeomorphism on the linear frame bundle $\mathcal{F}r(P_0)$ of P_0 , defined by

$$\widehat{f}_h(u_{\#}) = f_{h*} \circ u_{\#} \circ \text{Ad}_{h^{-1}} : \mathfrak{m} + \mathfrak{h}^0 \longrightarrow T_{f_h(y)}P^0$$

for any $u_{\#} \in \mathcal{F}r(P_0)|_y$. Using the definition of the adapted frames, one can check that \widehat{f}_h maps $\mathcal{F}r_{1*}(P_0)$ into itself. Moreover, using the fact that \widehat{f}_h preserves also the distributions (5.6) and the partial complex structures of the tangent spaces, one can check that $\widehat{f}_h(\mathcal{F}r_1(P_0)) \subset \mathcal{F}r_1(P_0)$ and $\widehat{f}_h(\widetilde{\mathcal{F}r}_1(P_0)) \subset \widetilde{\mathcal{F}r}_1(P_0)$.

Now, from Lemma 5.4 and the fact that $(\partial^1)^{\perp} + \ker \partial^*|_{C^{\ell}(\mathfrak{m}, \mathfrak{g})}$ is $\text{ad}(\mathfrak{h}^0)$ -invariant, it follows that \widehat{f}_h maps strongly adapted frames into strongly

adapted frames, inducing an automorphism of $\tilde{f}_h : P^1 \rightarrow P^1$. This shows the existence of a right action of H^0 and, consequently, of a right action of a semidirect product $H^0 \ltimes H^1$ on P^1 , which acts transitively and freely on the fibers of $\pi^0 \circ \pi^1 : P^1 \rightarrow M$. The last claim can be checked using Lemmas 5.4 and 6.8 and the definition of the right actions of $H^0 \ltimes H^1$. \square

In analogy with Remark 5.3, if $u_\sharp, u'_\sharp : \mathfrak{m} + \mathfrak{h}^0 \rightarrow T_y P^0$ are two strongly adapted frames in the same equivalence class $[u_\sharp] \in P^1|_y$, they determine the same graded isomorphism $u, u' : \mathfrak{m} + \mathfrak{h}^0 \rightarrow \mathfrak{gr}(T_y P^0, \mathcal{F})$.

We conclude observing that, in analogy with §5.3, there is a natural flag of distributions on P^1 , given by

$$\mathcal{D}_{-1}^1 \stackrel{\text{def}}{=} (\pi_*^1)^{-1}(\mathcal{D}_{-1}^0), \quad \mathcal{D}_{(0|-1)}^1 \stackrel{\text{def}}{=} (\pi_*^1)^{-1}(\mathcal{D}_{(0|-1)}^0), \quad \mathcal{D}_{(0|0)}^1 \stackrel{\text{def}}{=} (\pi_*^1)^{-1}(\mathcal{D}_{(0|0)}^0)$$

and $\mathcal{D}_1^1 = (\pi_*^1)^{-1}(0) = T^{\text{Vert}} P^1$. These distributions and the CR structure of (M, \mathcal{D}, J) induce filtrations of type (4.1), (4.3) on tangent spaces

$$\mathcal{F} : \underset{=V_{-2}}{T_y P^1} \supset \underset{=V_{-1}}{\mathcal{D}_{-1}^1|_y} \supset \underset{=V_0}{\mathcal{D}_{(0|-1)}^1|_y} \supset \underset{=V_1}{\mathcal{D}_0^1|_y} \supset \{0\}, \quad (6.26)$$

$$\mathcal{F}_* : \underset{=V_{-2}}{T_y P^1} \supset \underset{=V_{-1}}{\mathcal{D}_{-1}^1|_y} \supset \underset{=V_0}{\mathcal{D}_{(0|-1)}^1|_y} \supsetneq \underset{=V_{(0|0)}}{\mathcal{D}_{(0|0)}^1|_y} \supset \underset{=V_1}{\mathcal{D}_1^1|_y} \supset \{0\} \quad (6.27)$$

endowed with a (unique up to equivalences) partial complex structure J . By construction, any graded space $\mathfrak{gr}(T_y P^1, \mathcal{F}_*)$ is isomorphic to $\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1$.

7. THE SECOND PROLONGATION OF THE TANAKA STRUCTURE P^0

7.1. Adapted frames and nicely adapted frames of P^1 .

Definition 7.1. Let $z \in P^1|_y$ be a point over $y = \pi^1(z) \in P^0$. A linear frame $u_\sharp : \mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1 \rightarrow T_z P^1$, adapted to the filtration and partial complex structure of $T_x P^1$, is called *adapted frame of P^1* if

- i) the restriction $u_\sharp|_{\mathfrak{h}^0 + \mathfrak{h}^1} : \mathfrak{h}^1 \rightarrow \mathcal{D}_{(0|0)}^1|_z$ coincides with the isomorphism given by the right action of $\mathfrak{h}^0 + \mathfrak{h}^1 = \text{Lie}(H^0 \ltimes H^1)$ on P^1 ;
- ii) the projected frame $\underline{u}_\sharp = \pi_*^1 \circ u_\sharp|_{\mathfrak{m} + \mathfrak{h}^0} : \mathfrak{m} + \mathfrak{h}^0 \rightarrow T_y P^0$ is in the equivalence class $z = [\underline{u}_\sharp] \in P^1$.

The collection of such frames is called *bundle of adapted frames of P^1* and is denoted by $\mathcal{F}r_{2*}(P^1)$. We denote by $\pi_{2*} : \mathcal{F}r_{2*}(P^1) \rightarrow P^1$ the natural projection.

By the remarks in §6.3.2, any linear frame of a fiber $\mathcal{F}r_{2*}(P^1)|_z$ is uniquely determined by the corresponding s.h.s. $(H^{-2}, H^{-1}, H^{(0|-1)}, H^{(0|0)})$ of $V = T_z P^1$. From this and usual arguments (see also Remark 4.2), it follows that:

Lemma 7.2. *The triple $(\mathcal{F}r_{2*}(P^1), P^1, \pi_{2*})$ is a principal bundle over P^1 , with structure group $GL_{2*}(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$.*

In analogy with §6.2, for a given smooth field \mathcal{H} of adapted s.h.s. in the tangent spaces of a neighbourhood of $z \in P^1$, we may consider the *torsion of \mathcal{H} at z*

$$\tau_{\mathcal{H},z} \in \text{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1), \quad \tau_{\mathcal{H},z}(X, Y) = \widehat{\mathcal{H}}_z^{-1}([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_z), \quad (7.1)$$

the associated *c-torsion* $c_{\mathcal{H},z} = \tau_{\mathcal{H},z}|_{\Lambda^2(\mathfrak{m}^{-2} + \mathfrak{m}^{-1})}$ and the graded components $\tau_{\mathcal{H},z}^k$ and $c_{\mathcal{H},z}^k$. Besides this, we need to consider also the following definition.

Definition 7.3. Given a local field \mathcal{H} of adapted s.h.s. on a neighbourhood of z , we call

- *γ -torsion at z* the restriction $\gamma_{\mathcal{H},z} = \tau_{\mathcal{H},z}^1|_{\mathfrak{m}^{-1} \times \mathfrak{m}^0} \in \text{Hom}(\mathfrak{m}^{-1} \times \mathfrak{m}^0, \mathfrak{m}^0 + \mathfrak{h}^0)$.
- *δ -torsion at z* the linear map $\delta_{\mathcal{H},z} \in \text{Hom}(\mathfrak{m}^{-2} \times \mathfrak{m}^{0(10)}, \mathbb{R}E_2^0)$, defined by

$$\delta_{\mathcal{H},z} = \text{Im} \left(\left(\tau_{\mathcal{H},z}^2|_{\mathfrak{m}^{-2} \times \mathfrak{m}^0} \right)_{\mathbb{C}E_2^0} \right), \quad (7.2)$$

where $(\dots)_{\mathbb{C}E_2^0}$ denotes the projection onto $\mathbb{C}E_2^0 \subset \mathfrak{h}^0$.

Lemma 7.4. *The γ -torsion $\gamma_{\mathcal{H},z}$ depends only on $H = \mathcal{H}|_z$ in $T_z P^1$ and can be considered as a tensor γ_H associated with \widehat{H} . The subset $\mathcal{F}r_2(P^1) \subset \mathcal{F}r_{2*}(P^1)$ of adapted frames \widehat{H} such that*

$$\left(\gamma_H(e^{-1(10)}, e^{0(10)}) \right)_{\mathfrak{m}^{0(10)}} = 0 = \left(\gamma_H(e^{-1(01)}, e^{0(10)}) \right)_{\mathfrak{m}^{0(01)}} \quad (7.3)$$

is a reduction with structure group $\text{GL}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$.

Proof. The proof of the first claim is the perfect analogue of the argument used for Lemma 6.4. Also the second claim is proved in a very similar way. In fact, consider a fixed s.h.s. H_o in $T_z P^1$ and a field of s.h.s. \mathcal{H}_o on a neighbourhood \mathcal{U} of z with $\mathcal{H}_o|_z = H_o$. Any other field of s.h.s. \mathcal{H} is such that $\widehat{\mathcal{H}}|_{z'} = \widehat{\mathcal{H}}|_{z'} \circ A_{z'}$, $z' \in \mathcal{U}$, for some map $A = I + B$ with values in $\text{GL}_{2*}(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$. By usual arguments, we find that $\gamma_H(X, Y)$, with $H = \mathcal{H}_z$, $X \in \mathfrak{m}^{-1}$, $Y \in \mathfrak{m}^0$, is equal to

$$\gamma_H(X, Y) = \gamma_{H_o}(X, Y) + ([X, B_z(Y)])_{\mathfrak{m}^0}. \quad (7.4)$$

On the other hand, modulo terms of higher grades, the map B_z is of the form

$$B_z(e^{-2}) = \lambda e^{0(10)} + \bar{\lambda} e^{0(01)}, \quad B_z(e^{-1(10)}) = \mu E^{1(10)} + \mu' E^{1(01)},$$

$$B_z(e^{0(10)}) = \nu E^{1(10)} + \nu' E^{1(01)} \quad \text{for some } \lambda, \mu, \mu', \nu, \nu' \in \mathbb{C},$$

from which it follows that

$$\gamma_H(e^{-1(10)}, e^{0(10)}) = \gamma_{H_o}(e^{-1(10)}, e^{0(10)}) - \frac{1}{2} \nu e^{0(10)} - \frac{1}{2} \nu' E^{0(10)},$$

$$\gamma_H(e^{-1(01)}, e^{0(10)}) = \gamma_{H_o}(e^{-1(10)}, e^{0(10)}) - \frac{1}{2} \nu E^{0(01)} - \frac{1}{2} \nu' e^{0(01)}.$$

Hence, there always exist ν, ν' such that \mathcal{H} satisfies (7.3), so that $\mathcal{F}r_1(P^1)|_z$ is not empty. The same expressions show that $\mathcal{F}r_2(P^1)$ is a reduction with structure group $\mathrm{GL}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$. \square

The proof of next lemma is basically the same of Lemma 6.6 (iii).

Lemma 7.5. *Let \mathcal{H} be a field of s.h.s. on a neighbourhood of z , with associated adapted frames in the reduction $\mathcal{F}r_2(P^1)$. Then $c_{\mathcal{H},z}^2$ depends only on $H = \mathcal{H}|_z$ and it can be considered as a tensor c_H^2 , associated with $\hat{H} \in \mathcal{F}r_2(P^1)|_z$.*

Now, as it was pointed out in §6.3.2, we have that

$$\mathfrak{gl}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1) \simeq C_2^1(\mathfrak{m}_-, \mathfrak{g}) , \quad \mathrm{Tor}^2(\mathfrak{m}) \simeq C_2^2(\mathfrak{m}_-, \mathfrak{g}) .$$

So, by (6.20), we have that $\ker \partial^*|_{C_2^2(\mathfrak{m}_-, \mathfrak{g})}$ is complementary to $\partial \mathfrak{gl}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$ in $\mathrm{Tor}^2(\mathfrak{m}) \simeq C_2^2(\mathfrak{m}_-, \mathfrak{g})$ and it is invariant under $\mathrm{ad}(\mathfrak{h}^0 + \mathfrak{h}^1)$. We call *nicely adapted frame* of $T_z P^1$ any adapted frame $\hat{H} \in \mathcal{F}r_2(P^1)|_z$ whose c -torsion is such that

$$c_H^2 \in \ker \partial^*|_{C_2^2(\mathfrak{m}_-, \mathfrak{g})} . \quad (7.5)$$

Lemma 7.6. *The collection $\widetilde{\mathcal{F}r}_2(P^1) \subset \mathcal{F}r_2(P^1)$ of nicely adapted frames is a reduction with structure group $\widetilde{H}_{\sharp}^2 = \widetilde{H}^2 \ltimes \mathrm{GL}_3(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$, where $\widetilde{H}^2 \subset \mathrm{GL}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$ is the subgroup*

$$\widetilde{H}^2 = \{ I + B , B \in \mathfrak{gl}_2^{\mathrm{gr}}(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1) : \partial B = 0 \} .$$

The proof is identical to Lemma 6.8 and we omit it.

Lemma 7.7. *The real abelian Lie algebra $\widetilde{\mathfrak{h}}^2 = \mathrm{Lie}(\widetilde{H}^2)$ has dimension 2 and is generated by B_1, B_2 , defined by*

$$B_1(e^{-2}) = E^{0(10)} + E^{0(01)} , \quad B_1(e^{-1(10)}) = iE^{1(10)} , \quad B_1(e^{-1(01)}) = -iE^{1(01)} , \quad (7.6)$$

$$B_2(e^{-2}) = ie^{0(10)} - ie^{0(01)} , \quad B_2(e^{-1(10)}) = E^{1(01)} , \quad B_2(e^{-1(01)}) = E^{1(10)} . \quad (7.7)$$

Proof. Modulo terms of higher grades, a map $B \in \mathfrak{gl}_2(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$ is of the form

$$B(e^{-2}) = \lambda e^{0(10)} + \bar{\lambda} e^{0(01)} + \mu E^{0(10)} + \bar{\mu} E^{0(01)} , \\ B(e^{-1(10)}) = \nu E^{1(10)} + \nu' E^{1(01)} , \quad B(e^{-1(01)}) = \bar{\nu}' E^{1(10)} + \bar{\nu} E^{1(01)}$$

for some $\lambda, \mu, \nu, \nu' \in \mathbb{C}$. The condition $\partial B = 0$ is equivalent to

$$\left(B([e^{-2}, e^{-1(10)}]) \right)_{\mathfrak{m}^{-1}} = \left([B(e^{-2}), e^{-1(10)}] + [e^{-2}, B(e^{-1(10)})] \right)_{\mathfrak{m}^{-1}} , \\ \left(B([e^{-1(10)}, e^{-1(01)}]) \right)_{\mathfrak{m}^0 + \mathfrak{h}^0} = \left([B(e^{-1(10)}), e^{-1(01)}] + [e^{-1(10)}, B(e^{-1(01)})] \right)_{\mathfrak{m}^0 + \mathfrak{h}^0} .$$

A direct computation shows that this holds if and only if

$$i\mu = \nu , \quad i\bar{\mu} = \nu , \quad -i\bar{\lambda} = \nu' , \quad i\bar{\lambda} = \nu' ,$$

or, equivalently, if and only if $(\lambda, \mu, \nu, \nu') = (is, t, it, s)$ with $t, s \in \mathbb{R}$. The pairs $(t, s) = (1, 0)$, $(t, s) = (0, 1)$ give the generators (7.6) and (7.7). \square

7.2. Strongly adapted frames of P^1 and the second prolongation.

The next lemma is proved with the same line of arguments of Lemmas 6.4, 6.5 and 6.6. For brevity, we give only the statement.

Lemma 7.8. *Let \mathcal{H} be a field of s.h.s. on a neighbourhood of z , with associated adapted frames in the reduction $\widetilde{\mathcal{F}r}_2(P^1)$. The δ -torsion $\delta_{\mathcal{H},z}$ depends only on $H = \mathcal{H}|_z$ and it can be considered as a tensor δ_H , associated with $\widehat{H} \in \widetilde{\mathcal{F}r}_2(P^1)|_z$.*

Due to this, we may consider the following

Definition 7.9. We call *strongly adapted frame* of $T_z P^1$ any nicely adapted frame $\widehat{H} \in \widetilde{\mathcal{F}r}_2(P^1)|_z$ with a vanishing δ -torsion.

Lemma 7.10. *The subset $P^2_{\#} \subset \widetilde{\mathcal{F}r}_2(P^1)$ of strongly adapted frames is a reduction with structure group $H^2_{\#} = H^2 \ltimes \mathrm{GL}_3(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$ where $H^2 \subset \widetilde{H}^2_{\#}$ is the subgroup defined by*

$$H^2 = \{ I + B \text{ with } B \in \widetilde{\mathfrak{h}}^2 = \mathrm{Lie}(\widetilde{H}^2) \text{ such that } B(\mathfrak{m}^{-2}) \subset \mathfrak{h}^0 \} . \quad (7.8)$$

Proof. The proof is almost the same of the proof of Lemma 6.4, but there is a point that deserves a special care. Consider a fixed s.h.s. H_o in $T_z P^1$ and a field of nicely adapted s.h.s. \mathcal{H}_o on a neighbourhood \mathcal{U} of z with $\mathcal{H}_o|_z = H_o$. As in the other proofs, we recall that any other field of s.h.s. \mathcal{H} is such that $\widehat{\mathcal{H}}|_{z'} = \widehat{\mathcal{H}}_o|_{z'} \circ A_{z'}$, $z' \in \mathcal{U}$, for some map $A = I + B$ with values in $\widetilde{H}^2_{\#}$.

Let $\pi = \pi^0 \circ \pi^1 : P^1 \rightarrow M$ be the standard projection of P^1 onto M . As in the proof of Lemma 6.5, we observe that $\pi_* \left(\mathcal{D}^1_{(0|-1)} \right) = \mathcal{E}$ is an integrable complex distribution of dimension one, so that, with no loss of generality, we may assume that the field \mathcal{H}_o is such that

$$\pi_* \left([e_{\mathcal{H}_o}^{0(10)}, e_{\mathcal{H}_o}^{0(01)}] \right) = 0 . \quad (7.9)$$

Following the usual arguments, one finds that the value of $\delta_H(e^{-2}, e^{0(10)})$ is

$$\delta_H(e^{-2}, e^{0(10)}) = \delta_{H_o}(e^{-2}, e^{0(10)}) - \mathrm{Im} \widehat{\mathcal{H}}_o^{-1} \left([B(e^{-2})_{\mathcal{H}_o}, e^{0(10)}_{\mathcal{H}_o}]|_z \right)_{\mathbb{C}E_2^0} ,$$

where, according to Lemma 7.7, the map B is such that (modulo terms of higher grades)

$$\begin{aligned} B_{z'}(e^{-2}) &= is \left(e^{0(10)} - e^{0(01)} \right) + t \left(E^{0(10)} + E^{0(01)} \right) , \\ B_{z'}(e^{-1(10)}) &= -itE^{1(10)} + sE^{1(01)} , \\ B_{z'}(e^{-1(01)}) &= sE^{1(10)} + itE^{1(01)} , \quad B_z(e^{0(10)}) = B(e^{0(01)}) = 0 \end{aligned}$$

for some suitable real valued functions $s = s(z'), t = t(z') \in \mathbb{R}$. By definition of the vector fields $E_{\mathcal{H}_o}^{0(10)}, E_{\mathcal{H}_o}^{0(01)}$,

$$[B(e^{-2})_{\mathcal{H}_o}, e^{0(10)}_{\mathcal{H}_o}] \Big|_z = -is [e^{0(10)}_{\mathcal{H}_o}, e^{0(01)}_{\mathcal{H}_o}] \Big|_z .$$

We claim that $[e^{0(10)}_{\mathcal{H}_o}, e^{0(01)}_{\mathcal{H}_o}] \Big|_z$ has non trivial component along $i E_{2\mathcal{H}_o}^0 \Big|_z$. Suppose not. Then, using (7.9), for any $z' \in \mathcal{H}$,

$$[e^{0(10)}_{\mathcal{H}_o}, e^{0(01)}_{\mathcal{H}_o}] \Big|_{z'} = i\sigma(z')E_{1\mathcal{H}_o}^0 + i\tau(z')E_{2\mathcal{H}_o}^0 \bmod \mathcal{D}_1^{1\mathbb{C}}$$

for some *real* valued functions σ, τ such that $\tau(z) = 0$. This implies that

$$\left[[e^{0(10)}_{\mathcal{H}_o}, e^{0(01)}_{\mathcal{H}_o}], e^{-1(10)}_{\mathcal{H}_o} \right] \Big|_z = -i\sigma e^{-1(10)}_{\mathcal{H}_o} \Big|_z \bmod \mathcal{D}_{(0|-1)}^{1\mathbb{C}} . \quad (7.10)$$

On the other hand, by (6.7) and (6.16),

$$\begin{aligned} & \pi_*^1 \left(\left[[e^{0(10)}_{\mathcal{H}_o}, e^{0(01)}_{\mathcal{H}_o}], e^{-1(10)}_{\mathcal{H}_o} \right] \right) = \\ & = \pi_*^1 \left(\left[e^{0(10)}_{\mathcal{H}_o}, [e^{0(01)}_{\mathcal{H}_o}, e^{-1(10)}_{\mathcal{H}_o}] \right] \right) \bmod \mathcal{D}_{(0|-1)}^{0\mathbb{C}} = \\ & = -\pi_*^1 \left([e^{0(10)}_{\mathcal{H}_o}, e^{-1(01)}_{\mathcal{H}_o}] \right) \bmod \mathcal{D}_{(0|-1)}^{0\mathbb{C}} = \pi_*^1 \left(e^{-1(10)}_{\mathcal{H}_o} \right) \bmod \mathcal{D}_{(0|-1)}^{0\mathbb{C}} , \end{aligned} \quad (7.11)$$

which contradicts (7.10) and concludes the proof of the claim. Then

$$\delta_H(e^{-2}, e^{0(10)}) = \delta_{H_o}(e^{-2}, e^{0(10)}) - s\tau \quad \text{for some } 0 \neq \tau \in \mathbb{R} . \quad (7.12)$$

This shows that it is always possible to determine s such that $\delta_H(e^{-2}, e^{0(10)}) = 0$ and that H, H' correspond to strongly adapted frames at z if and only if they are related by a map B with $s = 0$. \square

By Lemmas 7.7 and 7.10, $Lie(H^2)$ is 1-dimensional and is generated by the map B_1 , described in (7.6). Notice that B_1 is equal to

$$B_1 = \text{ad}(E^2)|_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{m} + \mathfrak{h} + \mathfrak{h}^1 = \mathfrak{g}/\mathfrak{h}^2 , \quad (7.13)$$

and $Lie(H^2) \simeq \mathfrak{h}^2$. We can now define the *second prolongation of the Tanaka structure* P^0 : It is the quotient bundle

$$\pi^2 : P^2 = P_{\sharp}^2 / GL_3(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1) \longrightarrow P^1$$

where π^2 is the map induced by the natural projection $\pi_{\sharp}^2 : P_{\sharp}^2 \longrightarrow P^1$. It is a principal bundle over P^1 , but it also a principal bundle over M . In fact,

Lemma 7.11. *There exists a natural right action of a semidirect product $(H^0 \ltimes H^1) \ltimes H^2$ on the fiber bundle $\pi = \pi^0 \circ \pi^1 \circ \pi^2 : P^2 \longrightarrow M$, which makes (P^2, M, π) a principal bundle, canonically associated with the girdled CR structure of M . The Lie group $(H^0 \ltimes H^1) \ltimes H^2$ is isomorphic to the isotropy subgroup $H = \text{Aut}_{x_o}(M_o)$.*

Proof. For any $h \in H^0 \ltimes H^1$, let $f_h : P^1 \rightarrow P^1$ be the corresponding diffeomorphism, given by the right action, and by $\widehat{f}_h : \mathcal{F}r(P_1) \rightarrow \mathcal{F}r(P_1)$ the diffeomorphism of $\mathcal{F}r(P_1)$ defined by

$$\widehat{f}_h(u_\#) = f_{h*} \circ u_\# \circ \text{Ad}_h : \mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1 = \mathfrak{g}/\mathfrak{h}^2 \rightarrow T_{f_h(z)}P^1$$

for any $u_\# \in \mathcal{F}r(P_1)|_z$. Using definitions, one can see that \widehat{f}_h maps $\mathcal{F}r_{2*}(P_1)$ into itself and, since \widehat{f}_h preserves the flag of distributions and the partial complex structures of the tangent spaces, it is such that $\widehat{f}_h(\mathcal{F}r_2(P_1)) \subset \mathcal{F}r_2(P_1)$.

Using Lemmas 5.4, 6.9 and 6.10 and the fact that $\ker \partial^*|_{C^\ell(\mathfrak{m}_-, \mathfrak{g})}$ is $\text{ad}(\mathfrak{h}^0 + \mathfrak{h}^1)$ -invariant, we obtain that \widehat{f}_h preserves the bundle of strongly adapted frames $P_\#^2$ and induces an automorphism of the H^2 -bundle $\widetilde{f}_h : P^2 \rightarrow P^2$. This shows the existence of a right action of $H^0 \ltimes H^1$, which determines a right action of $(H^0 \ltimes H^1) \ltimes H^2$ on P^2 acting transitively and freely on the fibers of $\pi : P^2 \rightarrow M$.

The last claim is a consequence of the last claim of Lemma 6.10, of (7.13) and of the definition of the right action of $(H^0 \ltimes H^1) \ltimes H^2$. \square

In perfect analogy with P^1 , there exists a natural flag of distributions on P^2 , given by

$$\begin{aligned} \mathcal{D}_{-1}^2 &\stackrel{\text{def}}{=} (\pi_*^2)^{-1}(\mathcal{D}_{-1}^1), \quad \mathcal{D}_{(0|-1)}^2 \stackrel{\text{def}}{=} (\pi_*^2)^{-1}(\mathcal{D}_{(0|-1)}^1), \quad \mathcal{D}_{(0|0)}^2 \stackrel{\text{def}}{=} (\pi_*^2)^{-1}(\mathcal{D}_{(0|0)}^1) \\ \mathcal{D}_1^2 &\stackrel{\text{def}}{=} (\pi_*^2)^{-1}(\mathcal{D}_1^1), \quad \mathcal{D}_2^2 = (\pi_*^2)^{-1}(0) = T^{\text{Vert}}P^2. \end{aligned}$$

These distributions and the CR structure of (M, \mathcal{D}, J) determine filtrations of type (4.1), (4.3) and a partial complex structure J (determined up to equivalences) on each tangent space $T_w P^2$. All this makes any graded vector space $\mathfrak{gr}(T_w P^2, \mathcal{F}_*)$ isomorphic to the graded Lie algebra $\mathfrak{so}_{3,2}(\mathbb{R}) = \mathfrak{m} + \mathfrak{h}$.

8. THE CARTAN CONNECTION OF A GIRDLED CR MANIFOLD AND THE SOLUTION OF THE EQUIVALENCE PROBLEM

Following the same steps for the constructions of the first and second prolongations, we now consider the following

Definition 8.1. Let $w \in P^2|_z$ be a point over $z = \pi^2(w) \in P^1$. A linear frame $u_\# : \mathfrak{m} + \mathfrak{h} \rightarrow T_w P^2$, adapted to the filtration and partial complex structure of $T_w P^2$, is called *adapted frame of P^2* if

- i) the restriction $u_\#|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathcal{D}_{(0|0)}^2|_w$ coincides with the isomorphism determined by the right action of \mathfrak{h} on P^2 ;
- ii) the projected linear frame $\underline{u}_\# = \pi_*^2 \circ u_\#|_{\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1} : \mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1 \rightarrow T_z P^1$ is in the equivalence class $w = [\underline{u}_\#] \in P^2$.

The collection of such frames is called *bundle of adapted frames of P^2* and is denoted by $\mathcal{F}r_{3*}(P^2)$. We denote by $\pi_{3*} : \mathcal{F}r_{3*}(P^2) \rightarrow P^2$ the natural projection.

By construction, the frames in a fiber $\mathcal{F}r_3(P^1)|_w$ are completely determined by the corresponding s.h.s. $(H^{-2}, H^{-1}, H^{(0|-1)}, H^{(0|0)}, H^1)$ of $V = T_w P^2$. Moreover, from definitions, $(\mathcal{F}r_3(P^2), P^2, \pi_{3*})$ is a principal bundle over P^2 with structure group $\mathrm{GL}_{3*}(\mathfrak{m} + \mathfrak{h})$. For any smooth field \mathcal{H} of adapted s.h.s. on a neighbourhood of $w \in P^2$, we may consider the *torsion of \mathcal{H} at w*

$$\tau_{\mathcal{H},w} \in \mathrm{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{h}) , \quad \tau_{\mathcal{H},w}(X, Y) = \widehat{\mathcal{H}}_w^{-1}([X_{\mathcal{H}}, Y_{\mathcal{H}}]|_w) , \quad (8.1)$$

the *c-torsion* $c_{\mathcal{H},w} = \tau_{\mathcal{H},w}|_{\Lambda^2(\mathfrak{m}^{-2} + \mathfrak{m}^{-1})}$ and the following

Definition 8.2. Given a local field \mathcal{H} of adapted s.h.s. on a neighbourhood $\mathcal{U} \subset P^2$ of w , we call *ε -torsion* the restriction $\varepsilon_{\mathcal{H},z} = \tau_{\mathcal{H},z}^2|_{\mathfrak{m}^{-2} \times \mathfrak{m}^0}$.

Lemma 8.3. *The ε -torsion $\varepsilon_{\mathcal{H},w}$ depends only on $H = \mathcal{H}|_w$ in $T_w P^2$ and can be considered as a tensor ε_H , associated with the frame \widehat{H} . The collection $\mathcal{F}r_3(P^2) \subset \mathcal{F}r_{3*}(P^2)$ of adapted frames such that*

$$\left(\varepsilon_H(e^{-2}, e^{0(10)}) \right)_{\mathfrak{h}^0} = 0 \quad (8.2)$$

is a subbundle with structure group $\mathrm{GL}_3(\mathfrak{m} + \mathfrak{h})$.

Proof. The proof of the first claim is exactly as in Lemma 6.4. For the second claim, as usual, consider a fixed s.h.s. H_o in $T_z P^1$ and a field of s.h.s. \mathcal{H}_o on a neighbourhood \mathcal{U} of w with $\mathcal{H}_o|_w = H_o$. Any other field of s.h.s. \mathcal{H} is such that $\widehat{\mathcal{H}}|_{w'} = \widehat{\mathcal{H}}|_w \circ A_{w'}$, $w' \in \mathcal{U}$, for some map $A = I + B$ with values in $\mathrm{GL}_{3*}(\mathfrak{m} + \mathfrak{h})$. We have that that $\varepsilon_H(X, Y)$, with $H = \mathcal{H}_z$, $X \in \mathfrak{m}^{-2}$, $Y \in \mathfrak{m}^{0(10)}$, is equal to

$$\varepsilon_H(X, Y) = \varepsilon_{H_o}(X, Y) + ([X, B_w(Y)])_{\mathfrak{m}^0} . \quad (8.3)$$

Modulo a term of higher grade, the map B_w is of the form

$$B_w(e^{-2}) = \lambda E^{1(10)} + \bar{\lambda} E^{1(01)} , \quad B_w(e^{-1(10)}) = \mu E^{2(10)} + \mu' E^{2(01)} , \\ B_w(e^{0(10)}) = \nu E^2 \quad \text{for some } \lambda, \mu, \mu', \nu \in \mathbb{C} .$$

From this, it follows that

$$\varepsilon_H(e^{-2}, e^{0(10)}) = \varepsilon_{H_o}(e^{-2}, e^{0(10)}) + \nu(E^{0(10)} + E^{0(01)}) ,$$

which can be used to infer that $\mathcal{F}r_3(P^2)|_w$ is not empty and that $\mathcal{F}r_3(P^2)$ is a reduction with structure group $\mathrm{GL}_3(\mathfrak{m} + \mathfrak{h})$. \square

The proof of the following lemma is basically the same of Lemma 6.6 (iii) and we omit it.

Lemma 8.4. *Let \mathcal{H} be a field of s.h.s. on a neighbourhood of w , with corresponding frames in $\mathcal{F}r_3(P^2)$. Then $c_{\mathcal{H},w}^3$ depends only on $H = \mathcal{H}|_w$ and can be considered as a tensor c_H^3 , associated with $\widehat{H} \in \mathcal{F}r_3(P^2)|_w$.*

As pointed before,

$$\mathfrak{gl}_3(\mathfrak{m} + \mathfrak{h}) \simeq C_3^1(\mathfrak{m}_-, \mathfrak{g}) , \quad \text{Tor}^3(\mathfrak{m}) \simeq C_3^2(\mathfrak{m}_-, \mathfrak{g}) ,$$

so that $\ker \partial^*|_{C_3^2(\mathfrak{m}_-, \mathfrak{g})}$ is complementary to $\partial \mathfrak{gl}_3(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1)$ in $\text{Tor}^3(\mathfrak{m}) \simeq C_3^2(\mathfrak{m}_-, \mathfrak{g})$ and invariant under $\text{ad}(\mathfrak{h})$. We call *strongly adapted frame* of $T_w P^2$ any adapted frame $\hat{H} \in \mathcal{Fr}_3(P^2)|_w$ with

$$c_H^3 \in \ker \partial^*|_{C_3^2(\mathfrak{m}_-, \mathfrak{g})} . \quad (8.4)$$

The proof of next lemma is essentially the same of Lemmas 6.8 and 7.11.

Lemma 8.5. *The subset $P_\#^3 \subset \mathcal{Fr}_2(P^1)$ of strongly adapted frames is a reduction with structure group $H_\#^3 = H^3 \ltimes \text{GL}_4(\mathfrak{m} + \mathfrak{h})$, where*

$$H^3 = \{ I + B , \ B \in \mathfrak{gl}_3^{\text{gr}}(\mathfrak{m} + \mathfrak{h}^0 + \mathfrak{h}^1) : \partial B = 0 \} .$$

Moreover, there exists a natural right action of a semidirect product

$$((H^0 \ltimes H^1) \ltimes H^2) \ltimes H^3$$

on the fiber bundle $\pi = \pi^0 \circ \pi^1 \circ \pi^2 \circ \pi^3 : P^3 = P^3/\text{GL}_4(\mathfrak{m} + \mathfrak{h}) \longrightarrow M$, which makes (P^3, M, π) a principal bundle, canonically associated with the girdled CR structure of M .

Lemma 8.6. *The Lie algebra $\text{Lie}(H^3)$ is trivial and the structure group $\pi_\#^3 : P_\#^3 \longrightarrow P^2$ is isomorphic to $\text{GL}_4(\mathfrak{m} + \mathfrak{h})$. In particular, the principal bundle (P^3, M, π) has structure group isomorphic to $\text{Aut}_{x_o}(M_o)$ and it is H -equivalent to (P^2, M, π) .*

Proof. Recall that an element $B \in \mathfrak{gl}_3(\mathfrak{m} + \mathfrak{h})$ is (modulo a single term of higher grade) such that

$$B(e^{-2}) = \lambda E^{1(10)} + \bar{\lambda} E^{1(01)} , \quad B(e^{-1(10)}) = \mu E^2 , \quad B(e^{-1(01)}) = \bar{\mu} E^2$$

for some $\lambda, \mu \in \mathbb{C}$. The condition $\partial B = 0$ is equivalent to

$$\left(B([e^{-2}, e^{-1(10)}]) \right)_{\mathfrak{m}^0 + \mathfrak{h}^0} = \left([B(e^{-2}), e^{-1(10)}] + [e^{-2}, B(e^{-1(10)})] \right)_{\mathfrak{m}^0 + \mathfrak{h}^0} ,$$

which implies that

$$0 = \frac{\lambda}{2} e^{0(10)} + \frac{\bar{\lambda}}{2} E^{0(10)} - \mu E^{0(10)} - \mu E^{0(01)} ,$$

i.e., $\lambda = 0 = \mu$. From this, the claim follows. \square

We can now prove the main theorem.

Theorem 8.7. *There exists a Cartan connection $\omega : P^2 \longrightarrow \mathfrak{so}_{3,2}(\mathbb{R})$ on the principal bundle (P^2, M, π) , which is canonically associated with the girdled CR structure, i.e., satisfies the following two properties:*

- a) *for any local CR diffeomorphism $f : \mathcal{U} \subset M \longrightarrow M$, the naturally associated lifted map $\tilde{f} : P^2 \longrightarrow P^2$ is such that $\tilde{f}^* \omega = \omega$;*

- b) if $F : \mathcal{V} \subset P^2 \longrightarrow P^2$ is a local diffeomorphism such that $F^*\omega = \omega$, then $F = \tilde{f}|_{\mathcal{V}}$ for some lifted map \tilde{f} of a local CR diffeomorphism f of M .

Proof. By usual arguments, we may consider a bundle $\pi_4 : \mathcal{F}r_4(P^3) \longrightarrow P_3$, given by the frames of P_3 , defined in complete analogy with the bundle described in Definition 8.1. For any field \mathcal{H} of s.h.s., associated with frames in $\mathcal{F}r_4(P^3)$, the usual arguments show that the 4-order component $c_{\mathcal{H},w}^4$ of the c -torsion of \mathcal{H} at $w \in P^3 \simeq P^2$ depends just on $H = \mathcal{H}|_w$ and we may consider the reduction of $P_{\sharp}^4 \subset \mathcal{F}r_4(P^3)$ given by the frames with $\partial^* c_H^4 = 0$. As in previous proofs and by the fact that $\mathfrak{gl}_5(\mathfrak{m} + \mathfrak{h}) = 0$, for any $w \in P^3 (\simeq P^2)$, two frames in $P^4|_w$ are always related by an endomorphism $A = I + B$ with $B \in \mathfrak{gl}_4^{\text{gr}}(\mathfrak{m} + \mathfrak{h})$ such that $\partial B = 0$. A simple check shows that B is trivial, so that any fiber of $\pi_{4\sharp} : P_{\sharp}^4 \longrightarrow P^3$ contains exactly one element. This means that there exists a unique section

$$\sigma : P^3 \longrightarrow \mathcal{F}r_4(P^3) \quad \text{satisfying} \quad \partial^* c_{\sigma}^4 \equiv 0 \quad (8.5)$$

(i.e. with $\sigma_w \in P_{\sharp}^4|_w$ for any $w \in P^3$) .

Since, by Lemma 8.6, the H -bundle $\pi : P^2 \longrightarrow M$ can be identified with the H -bundle $\pi : P^3 \longrightarrow M$, we may consider the \mathfrak{g} -valued 1-form ω on $P^2 (\simeq P^3)$, defined by

$$\omega_w = (\sigma_w)^{-1} : T_w P^2 \simeq T_w P^3 \longrightarrow \mathfrak{m} + \mathfrak{h}$$

at any $w \in P^2 \simeq P^3$. We claim that ω is a Cartan connection. In fact, by definitions, for any $w \in P^2 \simeq P^3$ the linear map

$$\omega_w|_{\mathfrak{h}} = (\sigma_w)^{-1}|_{\mathfrak{h}} : \mathfrak{h} \longrightarrow T_w^{\text{Vert}} P^3 \simeq T_w^{\text{Vert}} P^2$$

coincides with the natural isomorphism between $\mathfrak{h} = \text{Lie}(H)$ and $T_w^{\text{Vert}} P^2$, determined by the right action of H on the fibers. Moreover, by the same arguments of Lemmas 6.10 and 7.11, the bundle of linear frames P_{\sharp}^4 is invariant under a right action of H on $P^3 \simeq P^2$, so that, for any $h \in H$ and $w \in P^3 \simeq P^2$,

$$R_{h*} \circ \sigma_w \circ \text{Ad}_h = \sigma_{w \cdot h} \quad \Longleftrightarrow \quad (R_h^* \omega)|_w = \text{Ad}_{h^{-1}} \cdot \omega_w ,$$

proving that ω is a Cartan connection modelled on $M_o = \text{SO}_{3,2}^0/H$.

To check (a), we recall that, by construction, the bundle $\pi^0 : P^0 \longrightarrow M$ and the bundles $\pi^{i+1} : P^{i+1} \longrightarrow P^i$, $0 \leq i \leq 3$, are quotients of bundles of linear frames of the underlying manifolds. Using just the definitions, one can check that the differential f_* of a CR diffeomorphism f of M maps the frames in P_{\sharp}^0 into frames in P_{\sharp}^0 and that the quotient $P^0 = P_{\sharp}^0/\text{GL}_1(\mathfrak{m}, J)$ is mapped into itself. This defines a canonical lift $f_* : P^0 \longrightarrow P^0$ on P^0 . Similarly, the differential $(f_*)_*$ induces a lift $(f_*)_* : P^1 \longrightarrow P^1$ on P^1 , the differential $((f_*)_*)_*$ determines a lift on P^2 and so on. In particular, we

obtain a natural lift $\tilde{f} : P^3(\simeq P^2) \longrightarrow P^3$, which preserves the unique section $\sigma : P^3 \longrightarrow P^4_\#$ and, consequently, the Cartan connection ω .

For (b), consider the basis $\mathcal{B} = (e_j^i, E_m^\ell)$ of $\mathfrak{so}_{3,2} = \mathfrak{m} + \mathfrak{h}$ introduced in §3.2 and the vector fields $(\hat{e}_j^i, \hat{E}_m^\ell)$ defined by

$$\hat{e}_j^i|_w \stackrel{\text{def}}{=} (\omega_w)^{-1}(e_j^i), \quad \hat{E}_m^\ell|_w \stackrel{\text{def}}{=} (\omega_w)^{-1}(E_m^\ell) \quad \text{for any } w \in P^2.$$

Notice that, by the construction of ω , the flows of the vector fields \hat{E}_m^ℓ are $\Phi_t^{\hat{E}_m^\ell} = R_{\exp(tE_m^\ell)}$. Assume that $F : \mathcal{V} \subset P^2 \longrightarrow P^2$ is a local diffeomorphism such that $F^*\omega = \omega$ and hence such that

$$F_*(\hat{e}_j^i) = \tilde{e}_j^i, \quad F_*(\hat{E}_m^\ell) = \hat{E}_m^\ell. \quad (8.6)$$

It follows that $F \circ R_{\exp(tE_m^\ell)} = R_{\exp(tE_m^\ell)} \circ F$, $t \in \mathbb{R}$, and it induces a local diffeomorphism $f : \pi(\mathcal{V}) \subset M \longrightarrow M$ on M . Moreover, using the fact that

$$\mathcal{D}_x = \langle \pi_*(\hat{e}_i^{-1}|_w), \pi_*(\hat{e}_i^0|_w), i = 1, 2 \rangle, \quad x \in M, \quad w \in \pi^{-1}(x) \subset P^2,$$

one gets that f is a local CR diffeomorphism. Finally, if we denote by $\tilde{f} : P^2|_{\pi(\mathcal{V})} \longrightarrow P^2$ the natural lift of f on P^2 , the map $F' = \tilde{f}^{-1} \circ F : \mathcal{V} \longrightarrow P^2$

- is a local diffeomorphism mapping the fields \hat{e}_j^i and \hat{E}_m^ℓ into themselves;
- induces the identity map $\text{Id}_{\pi(\mathcal{V})}$ on $\pi(\mathcal{V}) \subset M$.

By the properties of Cartan connections (use e.g. normal coordinates – see [SS1]), this occurs if and only if $F'(w) = w$ for any $w \in \mathcal{V}$, i.e. $F = \tilde{f}|_{\mathcal{V}}$. \square

Let ω be the Cartan connection introduced in previous theorem and

$$\vartheta^{-2}, \vartheta^{-1(10)}, \vartheta^{-1(01)}, \vartheta^{0(10)}, \vartheta^{0(01)}, \omega^{0(10)}, \omega^{0(01)}, \omega^{1(10)}, \omega^{1(01)}, \omega^2$$

the (\mathbb{R} - and \mathbb{C} -valued) 1-forms of P^2 that for any vector field X of P^2 give the components of the elements $\omega(X) \in \mathfrak{so}_{3,2}$ w.r.t. the basis formed by the matrices $e^{-2}, e^{-i(10)}, E^{i(10)}, E^2$ and their complex conjugates. Using the construction of ω , one can check that they satisfy the structure equations

$$\begin{aligned} d\vartheta^{-2} + \frac{i}{2}\vartheta^{-1(10)} \wedge \vartheta^{-1(01)} - \left(\omega^{0(10)} + \omega^{0(01)}\right) \wedge \vartheta^{-2} &= \Theta^{-2}, \\ d\vartheta^{-1(10)} - \vartheta^{0(10)} \wedge \vartheta^{-1(01)} - \left(\omega^{0(10)} + \omega^{0(01)}\right) \wedge \vartheta^{-1(10)} + i\omega^{1(10)} \wedge \vartheta^{-2} &= \Theta^{-1(10)}, \\ d\vartheta^{0(10)} - \left(\omega^{0(10)} - \omega^{0(01)}\right) \wedge \vartheta^{0(10)} + \frac{1}{2}\omega^{1(10)} \wedge \vartheta^{-1(10)} &= \Theta^{0(10)}, \\ d\omega^{0(10)} + \frac{1}{2}\omega^{1(01)} \wedge \vartheta^{-1(10)} + \omega^2 \wedge \vartheta^{-2} &= \Omega^{0(10)}, \\ d\omega^{1(10)} - \omega^{1(01)} \wedge \vartheta^{0(10)} + \omega^{1(10)} \wedge \omega^{0(01)} + i\omega^2 \wedge \vartheta^{-1(10)} &= \Omega^{1(10)}, \\ d\omega^2 - \frac{i}{2}\omega^{1(10)} \wedge \omega^{1(01)} + \left(\omega^{0(10)} + \omega^{0(01)}\right) \wedge \omega^2 &= \Omega^2, \end{aligned} \quad (8.7)$$

where the 2-forms Θ^α and Ω^a are of the form (here α, β, γ denote indices as $-2, -1(10)$ etc., and a, b, c denote indices as $2, 1(10)$, etc.):

$$\Theta^\alpha = \sum_{\beta, \gamma} T_{\beta|\gamma}^\alpha \vartheta^\beta \wedge \vartheta^\gamma, \quad \Omega^a = \sum_{b, c} R_{b|c}^a \vartheta^b \wedge \vartheta^c$$

with smooth functions $T_{\beta|\gamma}^\alpha$ and $R_{b|c}^a$, called *structure functions*, which satisfy constraints, corresponding to the conditions considered in Lemmas 6.4, 6.5, 6.8, 7.4, 7.6 and 7.10. For example, the condition considered in Lemma 6.4 implies that

$$T_{-1(10)|0(10)}^{-1(10)} = T_{-1(01)|0(10)}^{-1(01)} = 0.$$

Analogous constraints come from the other conditions: each of them either requires the vanishing of some structure function or imposes a linear relation between some of them.

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